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Generalizations of institutions and a forward morphism determined by the theorem of Herbrand-Schmidt-Wang

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Abstract. For **Sig**, the category of many-sorted signatures, and **Cat**, the category of $\boldsymbol{\mathcal{U}}$-categories, for a Grothendieck universe $\boldsymbol{\mathcal{U}}$, the structure consisting of: (1) the contravariant functor Alg from **Sig** to **Cat**, which sends $\mathbf{\Sigma} \in \mathbf{Sig}$ to $\mathbf{Alg}(\mathbf{\Sigma})$, the category of $\mathbf{\Sigma}$-algebras, (2) the pseudo-functor Ter from **Sig** to **Cat**, which sends $\mathbf{\Sigma} \in \mathbf{Sig}$ to $\mathbf{Ter}(\mathbf{\Sigma})$, the category of generalized terms for $\mathbf{\Sigma}$, (3) the family $\mathrm{Tr} = (\mathrm{Tr}^{\mathbf{\Sigma}})_{\mathbf{\Sigma} \in \mathbf{Sig}}$, where, for each $\mathbf{\Sigma} \in \mathbf{Sig}$, $\mathrm{Tr}^{\mathbf{\Sigma}}$ is the functor from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to **Set** that formalizes the realization of terms as term operations on algebras, and (4) the family $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \mathrm{Mor}(\mathbf{Sig})}$, where, for each morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ in **Sig**, $\theta^{\mathbf{d}}$ is a natural isomorphism between two suitable functors from $\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to **Set**, which shows the invariant character of the procedure of realization of terms under signature change, has led us to consider the following generalization of the concept of institution. Let **C** be a category, then an institution on **C** is a quadruple $(\mathbf{Sig}, \mathrm{Mod}, \mathrm{Sen}, (\alpha, \beta))$, where **Sig** is a category, $\mathrm{Mod}\colon \mathbf{Sig}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ and $\mathrm{Sen}\colon \mathbf{Sig} \longrightarrow \mathbf{Cat}$ are two pseudo-functors, and (α, β) is a pseudo-extranatural transformation from $\mathrm{Mod}(\cdot) \times \mathrm{Sen}(\cdot)$, the pseudo-functor obtained from Mod and Sen, to $\mathrm{K}_{\mathbf{C}}$, the functor which picks **C**. The richer structure of institutions accommodate "term-institutions" such as the motivating example integrating terms as "sentence" that are more basic and exhibit more structure than actual sentences built of terms. We have also defined a notion of 2-institution, parameterized by a given category **C**, which is roughly obtained by allowing the category **Sig** to be a 2-category, letting Mod and Ter be pseudo-functors, and (α, β) a pseudo-extranatural transformation from $\mathrm{Mod}(\cdot) \times \mathrm{Sen}(\cdot)$ to $\mathrm{K}_{\mathbf{C}}$. Now, a 2-category structure on **Sig** is not easy to motivate, namely it is not clear what the 2-cells should stand for. However, we have appropriately motivated this new concept by using the example of polyderivors between many-sorted signatures. Polyderivors yield on **Sig** a 2-category $\mathbf{Sig}_{\mathfrak{pd}}$ and make, in a natural way, Alg and Ter pseudo-functors and (Tr, θ) a pseudo-extranatural transformation, providing an example of 2-institution. From $\mathbf{Sig}_{\mathfrak{pd}}$ we obtain a 2-category $\mathbf{Spf}_{\mathfrak{pd}}$ of specifications and polyderivors, but additionally allowing the naturalness of the 2-cell to hold only up to provable equality, which provides another example of 2-institution. Finally, taking into account the theorem of Herbrand-Schmidt-Wang, we give a natural example of a forward morphism from the institution associated to the heterogeneous first-order logic to the institution associated to the ordinary first-order logic.

Keywords: Many-sorted algebra; generalized term; polyderivor; transformation of polyderivors; 2-institution on a category; theorem of Herbrand-Schmidt-Wang

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1. Introduction.

It seems appropriate to begin our work by recalling two direct ancestors of the concept of institution, as defined by Goguen and Burstall in [GoB84]. On the one hand, that (relatively forgotten) of *regular model-theoretic language* defined by Feferman in [Fef74], pp. 155–156, as a system $L = (\mathrm{Typ}_L, \mathrm{Str}_L, \mathrm{Stc}_L, \models_L)$ where Typ_L is a non-empty set of similarity types, called the *admitted types* of L, and Str_L, Stc_L, $\models_L$ are functions with domain Typ_L such that for each admitted types τ, τ':

(i) $\mathrm{Str}_L(\tau)$ is a sub-collection of $\mathrm{Str}(\tau)$, called the *admitted structures for* $L(\tau)$,

(ii) $\mathrm{Stc}_L(\tau)$ is a collection called the *sentences* of $L(\tau)$,

(iii) $\models_{L,\tau}$ is a sub-relation of $\mathrm{Str}_L(\tau) \times \mathrm{Stc}_L(\tau)$, called the *satisfaction* (or *truth*) *relation of* $L(\tau)$,

(iv) ***Expansion***. $\tau \subseteq \tau' \Rightarrow \mathrm{Stc}_L(\tau) \subseteq \mathrm{Stc}_L(\tau')$; $\mathfrak{M}' \in \mathrm{Str}_L(\tau') \Rightarrow \mathfrak{M}' \upharpoonright \tau \in \mathrm{Str}_L(\tau)$ and $\varphi \in \mathrm{Stc}_L(\tau) \Rightarrow [\mathfrak{M}' \upharpoonright \tau \models \varphi \Leftrightarrow \mathfrak{M}' \models \varphi]$,

(v) *Renaming*. Each $\tau \equiv_\gamma \tau'$ induces a $1-1$ correspondence $\overline{\gamma}\colon \mathrm{Stc}(\tau) \longrightarrow \mathrm{Stc}(\tau')$ such that if $\mathfrak{M} \in \mathrm{Str}_L(\tau)$ and $\mathfrak{M}' \in \mathrm{Str}(\tau')$ and $\mathfrak{M} \equiv_\gamma \mathfrak{M}'$, then $\mathfrak{M}' \in \mathrm{Str}_L(\tau')$ and $\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M}' \models \overline{\gamma}(\varphi)$, and

(vi) *Isomorphism*. If $\mathfrak{M} \in \mathrm{Str}_L(\tau)$ and $\mathfrak{M}' \in \mathrm{Str}(\tau)$ and $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M}' \in \mathrm{Str}_L(\tau)$ and $\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M}' \models \varphi$.

On the other hand, that of a *logic* $\mathfrak{L}^*$ defined by Barwise in [Bar74], pp. 234–235, where he says that it consists of a syntax and a semantics which fit together nicely. The *syntax of* $\mathfrak{L}^*$ is a functor $*$ on some category $\mathbf{C}$ of languages to the category of classes. The functor $*$ satisfies the following axiom:

> *Occurrence Axiom*. For every $\mathfrak{L}^*$-sentence φ there is a smallest (under $\subseteq$) language L_φ in $\mathbf{C}$ such that $\varphi \in L_\varphi^*$. If $i\colon L_\varphi \subseteq K$ is an inclusion morphism, so is $i^*\colon L_\varphi^* \subseteq K^*$.

The *semantics of* $\mathfrak{L}^*$ is a relation $\models$ such that if $\mathfrak{M} \models \varphi$, then $\mathfrak{M}$ is an L-structure for some L in $\mathbf{C}$ and $\varphi \in L^*$. It satisfies the following axiom:

> *Isomorphism Axiom*. If $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \cong \mathfrak{N}$, then $\mathfrak{N} \models \varphi$.

The syntax and semantics of $\mathfrak{L}^*$ fit together according to the final axiom.

> ***Translation Axiom***. For every $\mathfrak{L}^*$ sentence φ, every K-structure $\mathfrak{M}$ and every morphism $\alpha\colon L_\varphi \longrightarrow K$
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> $$\mathfrak{M} \models \alpha^*(\varphi) \text{ iff } \mathfrak{M}^{-\alpha} \text{ is an } L_\varphi\text{–structure and } \mathfrak{M}^{-\alpha} \models \varphi.$$

The theory of institutions of Goguen and Burstall, which arose within theoretical computer science, in response to the proliferation of logics in use there, is a categorial formalization of the *semantic* aspect of the intuitive notion of "logical system", and it has as objectives, according to Goguen and Burstall in [GoB86]: "(1) To support as much computer science as possible independently of the underlying logical system, (2) to facilitate the transfer of results (and artifacts such as theorem provers) from one logical system to another, and (3) to permit combining a number of different logical systems".

We recall that Goguen and Burstall in [GoB84], p. 229, define an *institution* as a category $\mathbf{Sign}$, of signatures, a functor Sen from $\mathbf{Sign}$ to $\mathbf{Set}$, giving the set of *sentences* over a given signature, a functor Mod from $\mathbf{Sign}$ to $\mathbf{Cat}^{\mathrm{op}}$, giving the category of *models* of a given signature, and, for each $\Sigma \in \mathbf{Sign}$, a satisfaction relation $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathrm{Sen}(\Sigma)$, where $|\cdot|$ is the endofunctor of $\mathbf{Cat}$ which sends a category to the discrete category on its set of objects, such that, for each morphism $\varphi\colon \Sigma \longrightarrow \Sigma'$, the

> ***Satisfaction Condition***. $\mathbf{M}' \models_{\Sigma'} \varphi(e)$ iff $\varphi(\mathbf{M}') \models_\Sigma e$,

holds for each $\mathbf{M}' \in |\mathbf{Mod}(\Sigma')|$ and each $e \in \mathrm{Sen}(\Sigma)$. Later on, in [GoB86], p. 316, they define an *institution* as a category $\mathbf{Sign}$, of signatures, a functor Sen from $\mathbf{Sign}$ to $\mathbf{Cat}$ (observe the large-scale change from $\mathbf{Set}$ to $\mathbf{Cat}$ in this definition, we emphasize), giving *sentences* and *proofs* over a given signature, a functor Mod from $\mathbf{Sign}$ to $\mathbf{Cat}^{\mathrm{op}}$, giving the category of *models* of a given signature, and a satisfaction relation $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times |\mathbf{Sen}(\Sigma)|$, for each $\Sigma \in |\mathbf{Sign}|$, such that

> ***Satisfaction Condition***: $\mathbf{M}' \models_{\Sigma'} \mathrm{Sen}(\varphi)s$ iff $\mathrm{Mod}(\varphi)\mathbf{M}' \models_\Sigma s$, for each $\varphi\colon \Sigma \longrightarrow \Sigma'$ in $\mathbf{Sign}$, $\mathbf{M}' \in |\mathbf{Mod}(\Sigma')|$ and $s \in |\mathbf{Sen}(\Sigma)|$, and
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> *Soundness Condition*: if $\mathbf{M} \models_\Sigma s$, then $\mathbf{M} \models_\Sigma s'$, for each $\mathbf{M} \in |\mathbf{Mod}(\Sigma)|$ and $s \longrightarrow s' \in \mathbf{Sen}(\Sigma)$.

Besides, the same authors, in [GoB86], p. 327, define, for a category $\mathbf{V}$, a *generalized* $\mathbf{V}$-*institution* as a pair of functors Mod, from $\mathbf{Sign}^{\mathrm{op}}$ to $\mathbf{Cat}$, and Sen, from $\mathbf{Sign}$ to $\mathbf{Cat}$, with an extranatural transformation $\models$ from $|\mathrm{Mod}(\cdot)| \times \mathrm{Sen}(\cdot)$ to $\mathbf{V}$. Observe that the second concept of institution falls under this last one because, taking as $\mathbf{V}$ the category $\mathbf{2}$, with two objects and just one morphism not the identity, the existence of an extranatural transformation from $|\mathrm{Mod}(\cdot)| \times \mathrm{Sen}(\cdot)$ to $\mathbf{2}$ is equivalent to the above satisfaction and soundness conditions.

But it happens that the compatibility of many-sorted terms and many-sorted algebras with respect to transformations between many-sorted signatures is also valid when many-sorted terms and many-sorted algebras (of *different* many-sorted signatures) are endowed with natural category structures, as we will prove in this article. From this fact it follows that the restriction imposed by Goguen and Burstall in [GoB86], p. 327, to the concept of institution, concretely, that the domain of the extranatural transformation $\models$ is $|\mathrm{Mod}(\cdot)| \times \mathrm{Sen}(\cdot)$, is a real loss of generality that prevents to reflect faithfully the involved complexity. This has led us to generalize the concept of institution towards two directions: (1) by parameterizing its "truth-value structure" by an arbitrary category, and (2) by allowing a 2-category structure on signatures, reflected in an appropriate way on the mappings Mod and Sen. The first direction of generalization integrates terms as "sentences" that are more basic and exhibit more structure than actual sentences, and the second allows for a very flexible notion of specification morphism and of equivalence between specifications. Specifically, we have defined a 2-institution on a given category $\mathbf{C}$ as a quadruple $(\mathbf{Sig}, \mathrm{Mod}, \mathrm{Sen}, (\alpha,\beta))$ consisting of a 2-*category* $\mathbf{Sig}$, a *pseudo-functor* $\mathrm{Mod}\colon \mathbf{Sig}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$, a *pseudo-functor* $\mathrm{Sen}\colon \mathbf{Sig} \longrightarrow \mathbf{Cat}$, and a *pseudo-extranatural* transformation $(\alpha,\beta)\colon\ \mathrm{Mod}(\cdot) \times \mathrm{Sen}(\cdot) \longrightarrow \mathrm{K}_{\mathbf{C}}$ from the pseudo-functor $\mathrm{Alg}(\cdot) \times \mathrm{Ter}(\cdot)$, obtained from Alg and Ter, to $\mathrm{K}_{\mathbf{C}}$, the functor which picks $\mathbf{C}$, both defined on $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ and taking values in $\mathbf{Cat}$. And an institution on $\mathbf{C}$ as a 2-institution for which $\mathbf{Sig}$ is an ordinary category.

On the other hand, as it is well-known, the theorem of Herbrand-Schmidt-Wang (see [Her30], [Sch38], and [Wan52]) about the reduction of heterogeneous first-order logic to homogeneous first-order logic, states that, for a heterogeneous first-order signature $\mathcal{S}$ and a heterogeneous $\mathcal{S}$-theory T, given a sentence of T and a proof for it in T, there is an effective way of finding a proof in T', the homogenization of T, for its translation in T'; and conversely, given a sentence of T' which has a translation in T, and given a proof for it in T', there is an effective way of finding a proof in T for its translation in T. On the basis of this theorem we have obtained, after accurately describing the institution Ht associated to heterogeneous first-order logic and the institution Ht associated to homogeneous first-order logic, a forward morphism from Ht to Hm founded on the domain unification.

Next we proceed to succinctly describe the contents of the following sections of this work.

The main goal of the *second section* is to construct the many-sorted term institution. To attain such a goal we begin by defining $\mathbf{MSet}$, the category of many-sorted sets and many-sorted mappings, $\mathbf{Sig}$, the category of standard many-sorted signatures, and $\mathbf{Alg}$, the category of standard many-sorted algebras and morphisms between many-sorted algebras of *different* many-sorted signatures, through the construction of Ehresmann-Grothendieck applied, respectively, to suitable contravariant functors MSet, Sig, and Alg. Then we remark that $\mathbf{MSet}$ and $\mathbf{Sig}$ are split bifibrations on $\mathbf{Set}$ and state that $\mathbf{MSet}$, $\mathbf{Sig}$, and $\mathbf{Alg}$ are bicomplete, that $\mathbf{Alg}$ is concrete, univocally transportable through a "forgetful"functor G into the fibered product $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, and that the functor G has a left adjoint $\mathbf{T}\colon \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} \longrightarrow \mathbf{Alg}$ which transforms objects of $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ into labelled term algebras in $\mathbf{Alg}$ and morphisms of $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ into translators between the associated labelled term algebras in $\mathbf{Alg}$. On the basis of the functor $\mathbf{T}$ we define, for every many-sorted signature $\boldsymbol{\Sigma}$, the category $\mathbf{Ter}(\boldsymbol{\Sigma})$, of generalized terms for $\boldsymbol{\Sigma}$, as the dual of the Kleisli category for $\mathbb{T}_{\boldsymbol{\Sigma}}$ (the standard monad derived from the adjunction between the category $\mathbf{Alg}(\boldsymbol{\Sigma})$, of $\boldsymbol{\Sigma}$-algebras, and the category $\mathbf{Set}^{S}$, of S-sorted set), and we extend this procedure to a *pseudo-functor* Ter from $\mathbf{Sig}$ to $\mathbf{Cat}$ which formalizes the procedure of translation for many-sorted terms. Then, to account exactly for the invariant character of the procedure of realization of the many-sorted terms in the many-sorted algebras, under change of many-sorted signature, we show that there exists a *pseudo-extranatural* transformation from a *pseudo-functor* obtained from Alg and Ter to the functor $\mathrm{K}_{\mathbf{Set}}$, both defined on $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ and taking values in the 2-*category* $\mathbf{Cat}$. Finally, after generalizing the concept of institution by means, essentially, of the notion of *pseudo-extranatural* transformation from a *pseudo-functor* to a constant functor, we get $\mathfrak{Tm}$, the many-sorted term institution on $\mathbf{Set}$.

In the *third section* we begin by defining, for a many-sorted signature $\boldsymbol{\Sigma}$, the concept of $\boldsymbol{\Sigma}$-equation, but for the generalized terms in the category $\mathbf{Ter}(\boldsymbol{\Sigma})$, the relation of satisfaction between many-sorted algebras and $\boldsymbol{\Sigma}$-equations, the consequence operator $\mathrm{Cn}_{\boldsymbol{\Sigma}}$, and by translating, for a morphism between many-sorted signatures, equations for the source many-sorted signature into equations for the target many-sorted signature. Then we continue with the proof of the satisfaction condition and, after defining a convenient pseudo-functor from $\mathbf{Sig}$ to $\mathbf{Cat}_{\boldsymbol{\mathcal{V}}}$, for an adequate Grothendieck universe $\boldsymbol{\mathcal{V}}$, we get $\mathfrak{LEq}$, the many-sorted equational institution on $\mathbf{2}$. Following this, after defining the category $\mathbf{Spf}$, of many-sorted specifications and many-sorted specification morphisms, we prove the existence of a contravariant functor, $\mathrm{Alg}^{\mathrm{sp}}$, and of a *pseudo-functor*, $\mathrm{Ter}^{\mathrm{sp}}$, from $\mathbf{Spf}$ to $\mathbf{Cat}$, that extend Alg and Ter, respectively. Then we state that from $\mathbf{Spf}^{\mathrm{op}} \times \mathbf{Spf}$ to the 2-category $\mathbf{Cat}$ there exists a *pseudo-functor*, obtained from $\mathrm{Alg}^{\mathrm{sp}}$ and $\mathrm{Ter}^{\mathrm{sp}}$, and a *pseudo-extranatural* transformation from it to the functor $\mathrm{K}_{\mathbf{Set}}$, and from this we get $\mathfrak{Spf}$, the many-sorted specification institution on $\mathbf{Set}$, and an institution morphism from $\mathfrak{Spf}$ to $\mathfrak{Tm}$ (actually, $\mathfrak{Tm}$ is a retract of $\mathfrak{Spf}$).

In the *fourth section* after defining the morphisms of Fujiwara between many-sorted signatures (that generalize the standard morphisms and the derivors between many-sorted signatures, as well as the families of basic mapping-formulas defined by Fujiwara in [Fuj59] for the single-sorted case), and the composition of these morphisms, we get the category $\mathbf{Sig}_{\mathfrak{pd}}$, of many-sorted signatures and morphisms of Fujiwara, and state that it can be obtained, up to isomorphism, as the Kleisli category for a monad in $\mathbf{Sig}$. Then we define a pseudo-functor (contravariant in the morphisms) $\mathrm{Alg}_{\mathfrak{pd}} \colon \mathbf{Sig}_{\mathfrak{pd}} \longrightarrow \mathbf{Cat}$ and, by applying the construction of Ehresmann-Grothendieck, we get a new category $\mathbf{Alg}_{\mathfrak{pd}}$, of many-sorted algebras and morphisms between many-sorted algebras which have, as a component, the morphisms of Fujiwara. Following this we define another pseudo-functor (covariant in the morphisms) $\mathrm{Ter}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$ which formalizes the procedure of translation for many-sorted terms, but now for the morphisms of Fujiwara. Then, to account exactly for the invariant character of the procedure of realization of the many-sorted terms in the many-sorted algebras, under change of many-sorted signature through the morphisms of Fujiwara, we show that there exists a pseudo-extranatural transformation from a pseudo-functor obtained from $\mathrm{Alg}_{\mathfrak{pd}}$ and $\mathrm{Ter}_{\mathfrak{pd}}$ to the functor $\mathrm{K}_{\mathbf{Set}}$, both defined on $\mathbf{Sig}_{\mathfrak{pd}}^{\mathrm{op}} \times \mathbf{Sig}_{\mathfrak{pd}}$ and taking values in the 2-category $\mathbf{Cat}$, and from this we get $\mathfrak{Tm}_{\mathfrak{pd}}$, the many-sorted term institution of Fujiwara.

In the *fifth section* we endow the category $\mathbf{Sig}_{\mathfrak{pd}}$ with a 2-category structure through the concept of transformation between morphisms of Fujiwara, which generalizes that one of equivalence between families of basic mapping-formulas, defined by Fujiwara in [Fuj60] for the single-sorted case. Then we prove that the transformations between morphisms of Fujiwara determine natural transformations between the functors associated to the morphisms of Fujiwara. From this we extend the pseudo-functors $\mathrm{Alg}_{\mathfrak{pd}}$ and $\mathrm{Ter}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$, and we get, in particular, by applying the construction of Ehresmann-Grothendieck to $\mathrm{Alg}_{\mathfrak{pd}}$, a corresponding 2-category $\mathbf{Alg}_{\mathfrak{pd}}$. Next, after proving that the transformations between morphisms of Fujiwara are compatible with the realization of the many-sorted terms in the many-sorted algebras, we show that there exists a pseudo-extranatural transformation from a pseudo-functor obtained from $\mathrm{Alg}_{\mathfrak{pd}}$ and $\mathrm{Ter}_{\mathfrak{pd}}$ to the functor $\mathrm{K}_{\mathbf{Set}}$, both defined on the 2-category $\mathbf{Sig}_{\mathfrak{pd}}^{\mathrm{op}} \times \mathbf{Sig}_{\mathfrak{pd}}$ and taking values in the 2-category $\mathbf{Cat}$, and from this we get $\mathfrak{Tm}_{\mathfrak{pd}}$, the many-sorted term 2-institution of Fujiwara.

In the *sixth section* we define a 2-category of specifications, $\mathbf{Spf}_{\mathfrak{pd}}$, with objects the specifications, 1-cells from a specification into a like one the polyderivors between the underlying signatures of the specifications that are compatible with the equations, and 2-cells from a 1-cell into a like one a convenient class of transformations between the polyderivors. Following this we state that the contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ and the pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$, both defined on the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$, can be lifted to the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$ as $\mathrm{Alg}_{\mathfrak{pd}}^{\mathrm{sp}}$ and $\mathrm{Ter}_{\mathfrak{pd}}^{\mathrm{sp}}$, respectively. Then we state that from the 2-category $\mathbf{Spf}_{\mathfrak{pd}}^{\mathrm{op}} \times \mathbf{Spf}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$ there exists a pseudo-functor, obtained from $\mathrm{Alg}_{\mathfrak{pd}}^{\mathrm{sp}}$ and $\mathrm{Ter}_{\mathfrak{pd}}^{\mathrm{sp}}$, and a pseudo-extranatural transformation from it to the functor $\mathrm{K}_{\mathbf{Set}}$, and from this we get $\mathfrak{Spf}_{\mathfrak{pd}}$, the many-sorted specification 2-institution of Fujiwara.

In the *seventh section*, taking into account the theorem of Herbrand-Schmidt-Wang, we provide a natural example of a forward morphism from the institution Ht to the institution Hm.

Every set we consider, unless otherwise stated, will be a $\boldsymbol{\mathcal{U}}$-small set or a $\boldsymbol{\mathcal{U}}$-large set, i.e., an element or a subset, respectively, of a Grothendieck universe $\boldsymbol{\mathcal{U}}$ (as defined, e.g., in [Mac98], p. 22), fixed once and for all. Besides, we agree that $\mathbf{Set}$ denotes the category which has as set of objects $\boldsymbol{\mathcal{U}}$ and as set of morphisms the subset of $\boldsymbol{\mathcal{U}}$ of all mappings between $\boldsymbol{\mathcal{U}}$-small sets, and, depending on the context, that $\mathbf{Cat}$ denotes either, the category of the $\boldsymbol{\mathcal{U}}$-categories (i.e., categories $\mathbf{C}$ such that the set of objects of $\mathbf{C}$ is a subset of the Grothendieck universe $\boldsymbol{\mathcal{U}}$, and the hom-sets of $\mathbf{C}$ elements of $\boldsymbol{\mathcal{U}}$), and functors between $\boldsymbol{\mathcal{U}}$-categories, or the 2-category of the $\boldsymbol{\mathcal{U}}$-categories, functors between $\boldsymbol{\mathcal{U}}$-categories, and natural transformations between functors.

In all that follows we use standard concepts and constructions from category theory, see e.g., [BoJ01], [Ehr65], [Gro71], [Kle65], and [Mac98]; classical universal algebra, see e.g., [Coh81], [Grä79]; categorical universal algebra, see e.g., [Bén68] and [Law63]; and many-sorted algebra, see e.g., [Bén68], [BiL48], [GoM85], [Hig63], and [Mat76]. Nevertheless, we have generically adopted the following notational and terminological conventions. For a set B, a family of sets $(A_i)_{i\in I}$, and a family of mappings $(f_i)_{i\in I}$ in $\prod_{i\in I}\mathrm{Hom}(B, A_i)$, we denote by $\langle f_i\rangle_{i\in I}$ the unique mapping from B to $\prod_{i\in I} A_i$ such that, for every $i \in I$, $f_i = \mathrm{pr}_i \circ \langle f_i\rangle_{i\in I}$, where pr_i is the canonical projection from $\prod_{i\in I} A_i$ to A_i. For a set S we agree upon denoting by $\mathbf{T}_\star(S) = (S^\star, \curlywedge, \lambda)$ the *free monoid on* S, where $S^\star$, the underlying set of $\mathbf{T}_\star(S)$, is $\bigcup_{n\in\mathbb{N}} S^n$, the set of all *words on* S, $\curlywedge$ the *concatenation* of words on S, and λ the *empty word* on S. For a word w on S, $|w|$ is the length of w. Moreover, $\mathbb{T}_\star = (\mathrm{T}_\star, \between, \curlywedge)$ is the standard monad in **Set** for the monoid specification, where, $\mathrm{T}_\star$ is the composition of the free monoid functor $\mathbf{T}_\star\colon \mathbf{Set} \longrightarrow \mathbf{Mon}$ and the forgetful functor $\mathrm{G}_{\mathbf{Mon}}\colon \mathbf{Mon} \longrightarrow \mathbf{Set}$, for every set S, $\between_S\colon S \longrightarrow S^\star$ the inclusion of S into $S^\star$, and $\curlywedge_S\colon S^{\star\star} \longrightarrow S^\star$ the merging of strings of words to words. To simplify the notation, we will write (s) instead of $\between_S(s)$. Furthermore, if $\varphi\colon S \longrightarrow T$ and $\psi\colon S \longrightarrow T^\star$ are mappings, then $\varphi^\star$ is the unique homomorphism from $\mathbf{T}_\star(S)$ to $\mathbf{T}_\star(T)$ such that $\varphi^\star\circ \between_S = \between_T \circ\varphi$, $\psi^\sharp\colon S^\star \longrightarrow T^\star$ the underlying mapping of the canonical extension of ψ to the free monoid $\mathbf{T}_\star(S)$ on S and $\psi^\star$ the unique monoid homomorphism from $\mathbf{T}_\star(S)$ to $\mathbf{T}_\star(T^\star)$ such that $\psi^\star\circ \between_S = \between_{T^\star} \circ\psi$. More specific notational conventions will be included and explained in the successive sections.

2. The many-sorted term institution.

Our main aim in this section is to show that the concept of "derived operation of an algebra", also known as "term operation of an algebra", elemental as it is, but fundamental for universal algebra, can be naturally subsumed under the notion of institution (see for this notion, e.g., [GoB86]), provided that an institution is meant not to be an extranatural transformation (as in [GoB86]) but a pseudo-extranatural transformation (as defined at the end of this section).

To attain the aim just mentioned we begin by a careful examination of the different types of things that are involved around it, namely many-sorted sets, signatures, algebras, terms, and generalized institutions. More specifically, in this section we define the category **MSet** of many-sorted sets, in which the many-sorted sets will be labelled with the sets of sorts, by applying the Ehresmann-Grothendieck's construction (henceforth abbreviated to EG-construction) (see [Ehr65], pp. 89–91 and [Gro71], pp. (sub.) 175–177) to a contravariant functor MSet from **Set** to **Cat**. Following this we define the categories **Sig**, of many-sorted signatures, and **Alg**, of many-sorted algebras, by applying also the EG-construction to suitable contravariant functors Sig from **Set** to **Cat**, and Alg from **Sig** to **Cat**, respectively.

Besides we prove the existence of a left adjoint **T** to a "forgetful" functor G from **Alg** to $\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$, and from this left adjoint **T** we define a pseudo-functor Ter from **Sig** to **Cat** which formalizes the procedure of translation for many-sorted terms.

Finally, to account exactly for the invariant character of the realization of many-sorted terms in many-sorted algebras under change of many-sorted signature, we prove the existence of a pseudo-extranatural transformation from a pseudo-functor on $\mathbf{Sig}^{\mathrm{op}}\times\mathbf{Sig}$ to **Cat**, induced by Alg and Ter, to the functor $K_{\mathbf{Set}}$, between the same categories, which is constantly **Set**. Then, after providing a generalization of the ordinary concept of institution, we prove that the pseudo-extranatural transformation is, in fact, part of an institution on **Set**, the so-called many-sorted term institution.

Before stating the first proposition of this section, we agree upon calling, henceforth, for a set (of sorts) $S \in \boldsymbol{\mathcal{U}}$, the objects of the category $\mathbf{Set}^S$ (i.e., the elements $A = (A_s)_{s\in S}$ of $\boldsymbol{\mathcal{U}}^S$) *S-sorted sets*; and the morphisms of the category $\mathbf{Set}^S$ from an S-sorted set A into another B (i.e., the ordered triples (A, f, B), abbreviated to $f\colon A \longrightarrow B$, where f is an element of $\prod_{s\in S}\mathrm{Hom}(A_s, B_s)$) *S-sorted mappings from A to B*. Furthermore, we also agree that a pseudo-functor F from a *category* **C** to a 2-*category* **D** consists of the following data:

1. An object mapping $F\colon \mathrm{Ob}(\mathbf{C}) \longrightarrow \mathrm{Ob}(\mathbf{D})$.
2. For every $x, y \in \mathbf{C}$, an hom-mapping $F\colon \mathrm{Hom}_{\mathbf{C}}(x, y) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(F(x), F(y))$.
3. For every morphisms $f\colon x \longrightarrow y$ and $g\colon y \longrightarrow z$ in **C**, an isomorphic 2-cell $\gamma^{f,g}$ from $F(g)\circ F(f)$ to $F(g\circ f)$.

4. For every $x \in \mathbf{C}$, an isomorphic 2-cell ν^x from $\mathrm{id}_{F(x)}$ to $F(\mathrm{id}_x)$.

These data must satisfy the following coherence axioms:

1. For morphisms $f\colon x \longrightarrow y$, $g\colon y \longrightarrow z$, and $h\colon z \longrightarrow t$ in $\mathbf{C}$,

$$\gamma^{g\circ f,h} \circ (\mathrm{id}_{F(h)} * \gamma^{f,g}) = \gamma^{f,h\circ g} \circ (\gamma^{g,h} * \mathrm{id}_{F(f)}).$$

2. For a morphism $f\colon x \longrightarrow y$ in $\mathbf{C}$,

$$\mathrm{id}_{F(f)} = \gamma^{\mathrm{id}_x,f} \circ (\mathrm{id}_{F(f)} * \nu^x) \text{ and } \mathrm{id}_{F(f)} = \gamma^{f,\mathrm{id}_y} \circ (\nu^y * \mathrm{id}_{F(f)}).$$

In the following proposition, that is basic for a great deal of what follows, for a mapping φ from S to T, we prove the existence of an adjunction $\coprod_\varphi \dashv \Delta_\varphi$ from the category of S-sorted sets to the category of T-sorted sets, as well as the existence of a contravariant functor MSet and of a pseudo-functor $\mathrm{MSet}^{\amalg}$ (related, respectively, to the right and left components of the adjunction) from **Set** to **Cat**.

Proposition 2.1. *Let $\varphi\colon S \longrightarrow T$ be a mapping. Then there are functors Δ_φ from $\mathbf{Set}^T$ to $\mathbf{Set}^S$ and $\coprod_\varphi$ from $\mathbf{Set}^S$ to $\mathbf{Set}^T$ such that $\coprod_\varphi \dashv \Delta_\varphi$. We write θ^φ, η^φ, and ε^φ, respectively, for the natural isomorphism, the unit, and the counit of the adjunction. Besides, there exists a contravariant functor MSet from **Set** to **Cat** which sends a set S to the category $\mathrm{MSet}(S) = \mathbf{Set}^S$, and a mapping φ from S to T to the functor Δ_φ from $\mathbf{Set}^T$ to $\mathbf{Set}^S$; and a pseudo-functor $\mathrm{MSet}^{\amalg}$ from **Set** to the 2-category **Cat** given by the following data*

1. *The object mapping of $\mathrm{MSet}^{\amalg}$ is that which sends a set S to the category $\mathrm{MSet}^{\amalg}(S) = \mathbf{Set}^S$.*
2. *The morphism mapping of $\mathrm{MSet}^{\amalg}$ is that which sends a mapping φ from S to T to the functor $\mathrm{MSet}^{\amalg}(\varphi) = \coprod_\varphi$ from $\mathbf{Set}^S$ to $\mathbf{Set}^T$.*
3. *For every $\varphi\colon S \longrightarrow T$ and $\psi\colon T \longrightarrow U$, the natural isomorphism $\gamma^{\varphi,\psi}$ from $\coprod_\psi \circ \coprod_\varphi$ to $\coprod_{\psi\circ\varphi}$ is that which is defined, for every S-sorted set A, as the U-sorted mapping that in the u-th coordinate is $((a,s),\varphi(s)) \mapsto (a,s)$, if there exists an $s \in S$ such that $u = \psi(\varphi(s))$, and is the identity at $\varnothing$, otherwise.*
4. *For every set S, the natural isomorphism ν^S from $\mathrm{Id}_{\mathbf{Set}^S}$ to $\coprod_{\mathrm{id}_S}$ is that which is defined, for every S-sorted set A and $s \in S$, as the canonical isomorphism from A_s to $A_s \times \{s\}$.*

Proof. Let Δ_φ be the functor from $\mathbf{Set}^T$ to $\mathbf{Set}^S$ defined as follows: its object mapping sends each T-sorted set A to the S-sorted set $A_\varphi = (A_{\varphi(s)})_{s\in S}$, i.e., the composite mapping $A \circ \varphi$; its arrow mapping sends each T-sorted mapping $f\colon A \longrightarrow B$ to the S-sorted mapping $f_\varphi = (f_{\varphi(s)})_{s\in S}\colon A_\varphi \longrightarrow B_\varphi$. Let $\coprod_\varphi$ be the functor from $\mathbf{Set}^S$ to $\mathbf{Set}^T$ defined as follows: its object mapping sends each S-sorted set A to the T-sorted set $\coprod_\varphi A = (\coprod_{s\in\varphi^{-1}[t]} A_s)_{t\in T}$; its arrow mapping sends each S-sorted mapping $f\colon A \longrightarrow B$ to the T-sorted mapping $\coprod_\varphi f = (\coprod_{s\in\varphi^{-1}[t]} f_s)_{t\in T}\colon \coprod_\varphi A \longrightarrow \coprod_\varphi B$. Then the functor $\coprod_\varphi$ is a left adjoint for Δ_φ since, for every S-sorted set A, $\coprod_\varphi A$ is $\mathrm{Lan}_\varphi A$, the left Kan extension of A along φ.

The proof that $\mathrm{MSet}^{\amalg}$ is a pseudo-functor follows easily from its definition and can therefore be left to the reader. □

Henceforth, when dealing with a pseudo-functor we will restrict ourselves to define explicitly only its object and morphism mappings, if about the remaining data and conditions there is not any doubt.

By applying the EG-construction to MSet we get the category of many-sorted sets as stated in the following definition.

Definition 2.2. The category **MSet**, of *many-sorted sets* and *many-sorted mappings*, is given by $\mathbf{MSet} = \int^{\mathbf{Set}} \mathrm{MSet}$. Therefore **MSet** has as objects the pairs (S, A), where S is a set and A an S-sorted set, and as morphisms from (S,A) to (T,B) the pairs (φ, f), where $\varphi\colon S \longrightarrow T$ and $f\colon A \longrightarrow B_\varphi$.

From the definition of the category **MSet** it follows, immediately, that the projection functor π_{MSet} for **MSet** is a split fibration. Furthermore, for every set S, the fiber of π_{MSet} at (S, id_S) is, essentially, the category $\mathbf{Set}^S$. On the other hand, if we apply the EG-construction to the pseudo-functor $\mathrm{MSet}^{\amalg}$, then we get a category with the same objects as **MSet**, but with morphisms from (S,A) to (T,B) the pairs

(φ, f), where $\varphi\colon S \longrightarrow T$ and $f\colon \coprod_{\varphi} A \longrightarrow B$. However, for every morphism $\varphi\colon S \longrightarrow T$, we have that $\coprod_{\varphi} \dashv \Delta_{\varphi}$, thus the categories $\int^{\mathbf{Set}} \mathrm{MSet}$ and $\int_{\mathbf{Set}} \mathrm{MSet}^{\amalg}$ are isomorphic (observe the use, in the symbol of *integration*, also called the *integral of Grothendieck*, of the subscript to indicate the covariant situation, and of the superscript to indicate the contravariant one). From this it follows that the functor π_{MSet} is also a split opfibration. Therefore we can assert that **MSet** is a split bifibration on **Set**.

Proposition 2.3. The category **MSet** is bicomplete.

Proof. This follows from Theorem 1, pp. 247–248, and Theorem 2, pp. 250–251, in [TBG91]. □

Our next goal is to define the category **Sig**. But before doing that we agree that, for a set of sorts S in $\boldsymbol{\mathcal{U}}$, $\mathbf{Sig}(S)$ denotes the category of S-sorted signatures and S-sorted signature morphisms, i.e., the category $\mathbf{Set}^{S^{\star} \times S}$, where $S^{\star}$ is the underlying set of the free monoid on S. Therefore an S*-sorted signature* is a function Σ from $S^{\star} \times S$ to $\boldsymbol{\mathcal{U}}$ which sends a pair $(w, s) \in S^{\star} \times S$ to the set $\Sigma_{w,s}$ of the *formal operations* of *arity* w, *sort* (or *coarity*) s, and *biarity* (w, s); and an S*-sorted signature morphism* from Σ to Σ' is an ordered triple (Σ, d, Σ'), written as $d\colon \Sigma \longrightarrow \Sigma'$, where $d = (d_{w,s})_{(w,s) \in S^{\star} \times S} \in \prod_{(w,s) \in S^{\star} \times S} \mathrm{Hom}(\Sigma_{w,s}, \Sigma'_{w,s})$. Thus, for every $(w, s) \in S^{\star} \times S$, $d_{w,s}$ is a mapping from $\Sigma_{w,s}$ to $\Sigma'_{w,s}$ which sends a formal operation σ in $\Sigma_{w,s}$ to the formal operation $d_{w,s}(\sigma)$ ($d(\sigma)$ for short) in $\Sigma'_{w,s}$.

Proposition 2.4. There exists a contravariant functor Sig from **Set** to **Cat**. Its object mapping sends each set of sorts S to $\mathrm{Sig}(S) = \mathbf{Sig}(S)$; its arrow mapping sends each mapping φ from S to T to the functor $\mathrm{Sig}(\varphi) = \Delta_{\varphi^{\star} \times \varphi}$ from $\mathbf{Sig}(T)$ to $\mathbf{Sig}(S)$ which relabels T-sorted signatures into S-sorted signatures, i.e., $\mathrm{Sig}(\varphi)$ assigns to a T-sorted signature Λ the S-sorted signature $\mathrm{Sig}(\varphi)(\Lambda) = \Lambda_{\varphi^{\star} \times \varphi}$, and assigns to a morphism of T-sorted signatures d from Λ to Λ' the morphism of S-sorted signatures $\mathrm{Sig}(\varphi)(d) = d_{\varphi^{\star} \times \varphi}$ from $\Lambda_{\varphi^{\star} \times \varphi}$ to $\Lambda'_{\varphi^{\star} \times \varphi}$.

By applying the EG-construction to Sig we get the category of many-sorted signatures as stated in the following definition.

Definition 2.5. The category **Sig**, of *many-sorted signatures* and *many-sorted signature morphisms*, is given by $\mathbf{Sig} = \int^{\mathbf{Set}} \mathrm{Sig}$. Therefore **Sig** has as objects the pairs (S, Σ), where S is a set of sorts and Σ an S-sorted signature and as many-sorted signature morphisms from (S, Σ) to (T, Λ) the pairs (φ, d), where $\varphi\colon S \longrightarrow T$ is a morphism in **Set** while $d\colon \Sigma \longrightarrow \Lambda_{\varphi^{\star} \times \varphi}$ is a morphism in $\mathbf{Sig}(S)$. The composition of $(\varphi, d)\colon (S, \Sigma) \longrightarrow (T, \Lambda)$ and $(\psi, e)\colon (T, \Lambda) \longrightarrow (U, \Omega)$, denoted by $(\psi, e) \circ (\varphi, d)$, is $(\psi \circ \varphi, e_{\varphi^{\star} \times \varphi} \circ d)$, where $e_{\varphi^{\star} \times \varphi}\colon \Lambda_{\varphi^{\star} \times \varphi} \longrightarrow (\Omega_{\psi^{\star} \times \psi})_{\varphi^{\star} \times \varphi} (= \Omega_{(\psi \circ \varphi)^{\star} \times (\psi \circ \varphi)})$. Henceforth, unless otherwise stated, we will write $\boldsymbol{\Sigma}$, $\boldsymbol{\Lambda}$, $\boldsymbol{\Omega}$, and $\boldsymbol{\Xi}$ instead of (S, Σ), (T, Λ), (U, Ω), and (X, Ξ), respectively, and $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{h}$, instead of (φ, d), (ψ, e), and (γ, h), respectively. Furthermore, to shorten terminology, we will say *signature* and *signature morphism* instead of *many-sorted signature* and *many-sorted signature morphism*, respectively.

Remark 1. P.J. Higgins in [Hig63] allows the variation of S but holds Σ fixed, while, J. Bénabou in [Bén68] follows precisely the inverse criterium.

The category **Sig**, as was the case for **MSet**, is also a split bifibration on **Set** through the projection functor π_{Sig} for **Sig**.

Since the category **Sig** can be identified to a subcategory of the category $\mathbf{Sig}_{\mathfrak{pd}}$, defined in the fourth section, we refer to that section for examples of signature morphisms.

Proposition 2.6. The category **Sig** is bicomplete.

Proof. This follows from Theorem 1, pp. 247–248, and Theorem 2, pp. 250–251, in [TBG91]. □

Since it will be used afterwards we introduce, for a signature $\boldsymbol{\Sigma}$, an S-sorted set A, an S-sorted mapping f from A to B, and a word w on S, i.e., an element w of $S^{\star}$, the following notation and terminology. We write $|w|$ for the length of the word w, A_w for $\prod_{i \in |w|} A_{w_i}$, and f_w for the mapping $\prod_{i \in |w|} f_{w_i}$ from A_w to B_w which sends $(a_i)_{i \in |w|}$ in A_w to $(f_{w_i}(a_i))_{i \in |w|}$ in B_w. Moreover, we let $\mathrm{HO}_S(A)$ stand for the $S^{\star} \times S$-sorted set $(\mathrm{Hom}(A_w, A_s))_{(w,s) \in S^{\star} \times S}$ and we call it the $S^{\star} \times S$-sorted set of the *finitary operations on* A.

We proceed next to define the category **Alg** of many-sorted algebras. But before doing that we agree that, for an arbitrary but fixed signature $\boldsymbol{\Sigma}$, $\mathbf{Alg}(\boldsymbol{\Sigma})$ denotes the category of $\boldsymbol{\Sigma}$-algebras (and $\boldsymbol{\Sigma}$-homomorphisms).

By a $\mathbf{\Sigma}$-*algebra* is meant a pair $\mathbf{A} = (A, F)$, where A is an S-sorted set and F a $\mathbf{\Sigma}$-*algebra structure on* A, i.e., a morphism $F = (F_{w,s})_{(w,s)\in S^{\star}\times S}$ in $\mathbf{Sig}(S)$ from Σ to $\mathrm{HO}_S(A)$ (for a pair $(w, s) \in S^{\star} \times S$ and a $\sigma \in \Sigma_{w,s}$, to simplify notation we let F_σ stand for $F_{w,s}(\sigma)$). A $\mathbf{\Sigma}$-*homomorphism* from a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ to another $\mathbf{B} = (B, G)$, is a triple $(\mathbf{A}, f, \mathbf{B})$, written as $f\colon \mathbf{A} \longrightarrow \mathbf{B}$, where f is an S-sorted mapping from A to B that preserves the structure in the sense that, for every (w, s) in $S^{\star} \times S$, every σ in $\Sigma_{w,s}$, and every $(a_i)_{i\in|w|}$ in A_w, it happens that $f_s(F_\sigma((a_i)_{i\in|w|})) = G_\sigma(f_w((a_i)_{i\in|w|}))$.

Proposition 2.7. There exists a contravariant functor Alg from $\mathbf{Sig}$ to $\mathbf{Cat}$. Its object mapping sends each signature $\mathbf{\Sigma}$ to $\mathrm{Alg}(\mathbf{\Sigma}) = \mathbf{Alg}(\mathbf{\Sigma})$, the category of $\mathbf{\Sigma}$-algebras; its arrow mapping sends each signature morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ to the functor $\mathrm{Alg}(\mathbf{d}) = \mathbf{d}^{*}\colon \mathbf{Alg}(\mathbf{\Lambda}) \longrightarrow \mathbf{Alg}(\mathbf{\Sigma})$ defined as follows: its object mapping sends each $\mathbf{\Lambda}$-algebra $\mathbf{B} = (B, G)$ to the $\mathbf{\Sigma}$-algebra $\mathbf{d}^{*}(\mathbf{B}) = (B_\varphi, G^{\mathbf{d}})$, where $G^{\mathbf{d}}$ is the composition of the $S^{\star}\times S$-sorted mappings d from Σ to $\Lambda_{\varphi^{\star}\times\varphi}$ and $G_{\varphi^{\star}\times\varphi}$ from $\Lambda_{\varphi^{\star}\times\varphi}$ to $\mathrm{HO}_T(B)_{\varphi^{\star}\times\varphi}$ (for $\sigma \in \Sigma_{w,s}$, to shorten notation, we let $G_{d(\sigma)}$ stand for the value of $G^{\mathbf{d}}$ at σ); its arrow mapping sends each $\mathbf{\Lambda}$-homomorphism f from $\mathbf{B}$ to $\mathbf{B}'$ to the $\mathbf{\Sigma}$-homomorphism $\mathbf{d}^{*}(f) = f_\varphi$ from $\mathbf{d}^{*}(\mathbf{B})$ to $\mathbf{d}^{*}(\mathbf{B}')$.

Proof. For every $\mathbf{\Lambda}$-algebra $\mathbf{B} = (B, G)$ it is the case that G is a morphism from Λ to $\mathrm{HO}_T(B)$. Then, by composing d and $G_{\varphi^{\star}\times\varphi}$, and taking into account that $\mathrm{HO}_T(B)_{\varphi^{\star}\times\varphi}$ is identical to $\mathrm{HO}_S(B_\varphi)$, we infer that $G^{\mathbf{d}} = G_{\varphi^{\star}\times\varphi} \circ d$ is a $\mathbf{\Sigma}$-algebra structure on the S-sorted set B_φ. On the other hand, for every (w, s) in $S^{\star} \times S$ and every $\sigma \in \Sigma_{w,s}$, it happens that $d(\sigma) \in \Lambda_{\varphi^{\star}(w),\varphi(s)}$. Thus, f being a $\mathbf{\Lambda}$-homomorphism from (B, G) to (B', G'), we infer that $f_{\varphi(s)} \circ G_{d(\sigma)} = G'_{d(\sigma)} \circ f_{\varphi^{\star}(w)}$. Hence, since $G^{\mathbf{d}}_\sigma = G_{d(\sigma)}$ and $G'^{\mathbf{d}}_\sigma = G'_{d(\sigma)}$, we have that $(f_\varphi)_s \circ G^{\mathbf{d}}_\sigma = G'^{\mathbf{d}}_\sigma \circ (f_\varphi)_w$. Therefore f_φ is a $\mathbf{\Sigma}$-homomorphism from $(B_\varphi, G^{\mathbf{d}})$ to $(B'_\varphi, G'^{\mathbf{d}})$.

Since identities and composites are, obviously, preserved by $\mathbf{d}^{*}$, it follows that $\mathbf{d}^{*}$ is a functor from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$. □

By applying the EG-construction to Alg we get the category of many-sorted algebras as stated in the following definition.

Definition 2.8. The category $\mathbf{Alg}$, of *many-sorted algebras* and *many-sorted algebra homomorphisms*, is given by $\mathbf{Alg} = \int^{\mathbf{Sig}} \mathrm{Alg}$. Therefore the category $\mathbf{Alg}$ has as objects the pairs $(\mathbf{\Sigma}, \mathbf{A})$, where $\mathbf{\Sigma}$ is a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, and as morphisms from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the pairs $(\mathbf{d}, f)$, with $\mathbf{d}$ a signature morphism from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and f a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^{*}(\mathbf{B})$. Henceforth, to shorten terminology, we will say *algebra* and *algebra homomorphism*, or, simply, *homomorphism*, instead of *many-sorted algebra* and *many-sorted algebra homomorphism*, respectively.

From the definition of $\mathbf{Alg}$ it follows that the projection functor π_{Alg} from $\mathbf{Alg}$ to $\mathbf{Sig}$ is a fibration. Moreover, for every set of sorts S, the fiber of $\pi_{\mathrm{Sig},\mathrm{Alg}} = \pi_{\mathrm{Sig}} \circ \pi_{\mathrm{Alg}}$ at (S, id_S) is, essentially, the category $\mathbf{Alg}(S)$ with objects the pairs $(\Sigma, \mathbf{A})$, where Σ is an S-sorted signature and $\mathbf{A} = (A, F)$ a Σ-algebra, and morphisms from $(\Sigma, \mathbf{A})$ to $(\Lambda, \mathbf{B})$, where $\mathbf{B} = (B, G)$, the pairs (d, f), where d is an S-sorted signature morphism from Σ to Λ and f a Σ-homomorphism from $\mathbf{A}$ to $\mathbf{B}^d = (B, G \circ d)$.

Since the category $\mathbf{Alg}$ can be identified to a subcategory of the category $\mathbf{Alg}_{\mathbf{p}\mathfrak{d}}$, defined in the fourth section, we refer to that section for examples of homomorphisms between algebras.

Proposition 2.9. The category $\mathbf{Alg}$ is a concrete and univocally transportable category.

Proof. It is enough to specify a functor from $\mathbf{Alg}$ to a convenient category of sorted sets labelled by signatures. Let $\mathrm{G}_{\mathbf{MSet}}$ be the forgetful functor from $\mathbf{Alg}$ to $\mathbf{MSet}$ (that is not a fibration), and $(\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}, (\mathrm{P}_0, \mathrm{P}_1))$ the pullback of the projection functors $\pi_{\mathrm{MSet}}\colon \mathbf{MSet} \longrightarrow \mathbf{Set}$ and $\pi_{\mathrm{Sig}}\colon \mathbf{Sig} \longrightarrow \mathbf{Set}$. Then we have that the structural functors P_0 and P_1 are fibrations, and that the unique functor G from $\mathbf{Alg}$ to $\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$ such that $\mathrm{P}_0 \circ \mathrm{G} = \mathrm{G}_{\mathbf{MSet}}$ and $\mathrm{P}_1 \circ \mathrm{G} = \pi_{\mathrm{Alg}}$ makes the category $\mathbf{Alg}$ a concrete and univocally transportable category on the category $\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$. □

Before we prove the existence of a left adjoint $\mathbf{T}$ to $\mathrm{G}\colon \mathbf{Alg} \longrightarrow \mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$, we agree on the following notation and terminology. For a signature $\mathbf{\Sigma}$ in $\mathbf{Sig}$, the functor $\mathbf{T}_{\mathbf{\Sigma}}$ from $\mathbf{Set}^S$ to $\mathbf{Alg}(\mathbf{\Sigma})$ is the left adjoint to the forgetful functor $\mathrm{G}_{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Set}^S$. For a signature $\mathbf{\Sigma}$ and an S-sorted set of variables X, $\mathbf{T}_{\mathbf{\Sigma}}(X)$ is the *free* (also called the *term* or *word*) $\mathbf{\Sigma}$-*algebra on* X, and η_X is the *insertion (of the generators)* X *into* $\mathrm{T}_{\mathbf{\Sigma}}(X)$, the underlying S-sorted set of $\mathbf{T}_{\mathbf{\Sigma}}(X)$. For a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ and a *valuation* f *of the* S*-sorted set of variables* X *in* A, i.e., an S-sorted mapping f from X to A, we will denote by $f^{\sharp}$ the *canonical* *extension of* f *to* $\mathbf{T}_{\mathbf{\Sigma}}(X)$, i.e., the unique $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{A}$ such that $f^{\sharp} \circ \eta_X = f$. For an S-sorted mapping f from X to Y, we will denote by $f^{@}$ the unique $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Sigma}}(Y)$ such that $f^{@} \circ \eta_X = \eta_Y \circ f$, i.e., the value of the functor $\mathbf{T}_{\mathbf{\Sigma}}$ at f. Therefore $f^{@}$ is also $(\eta_Y \circ f)^{\sharp}$. Moreover, by transposing to the many-sorted case the terminology coined for the single-sorted case, we call, for $s \in S$, the elements of $\mathrm{T}_{\mathbf{\Sigma}}(X)_s$, *many-sorted terms for* $\mathbf{\Sigma}$ *of type* (X, s), henceforth abbreviated to *terms for* $\mathbf{\Sigma}$ *of type* (X, s), or, simply, to *terms of type* (X, s).

Proposition 2.10. There exists a functor $\mathbf{T}\colon \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} \longrightarrow \mathbf{Alg}$ left adjoint to the functor G from $\mathbf{Alg}$ to $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$.

Proof. The functor $\mathbf{T}$ from $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ to $\mathbf{Alg}$ given on objects (S, Σ, X) by $\mathbf{T}(S, \Sigma, X) = (\mathbf{\Sigma}, \mathbf{T}_{\mathbf{\Sigma}}(X))$ and on arrows $(\varphi, d, f)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$ by $\mathbf{T}(\varphi, d, f) = (\mathbf{d}, f^{\mathbf{d}})\colon (\mathbf{\Sigma}, \mathbf{T}_{\mathbf{\Sigma}}(X)) \longrightarrow (\mathbf{\Lambda}, \mathbf{T}_{\mathbf{\Lambda}}(Y))$, where $f^{\mathbf{d}} = ((\eta_Y)_{\varphi} \circ f)^{\sharp}$ is the canonical extension of the S-sorted mapping $(\eta_Y)_{\varphi} \circ f$ from X to $\mathrm{T}_{\mathbf{\Lambda}}(Y)_{\varphi}$ to the free $\mathbf{\Sigma}$-algebra on X, is left adjoint to the functor G. □

For a morphism $(\varphi, d, f)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$ in $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, the functor $\mathbf{T}$ acting on (φ, d, f) allows us to get the $\mathbf{\Sigma}$-homomorphism $f^{\mathbf{d}}$ from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Lambda}}(Y)_{\varphi}$. Hence, for each $s \in S$, $f^{\mathbf{d}}_s$ translates terms for $\mathbf{\Sigma}$ of type (X, s) into terms for $\mathbf{\Lambda}$ of type $(Y, \varphi(s))$. In particular, the unit η^{φ} of the adjunction $\coprod_{\varphi} \dashv \Delta_{\varphi}$ provides, for each S-sorted set X, the S-sorted mapping $\eta^{\varphi}_X\colon X \longrightarrow (\coprod_{\varphi} X)_{\varphi}$ and if $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ is a morphism of signatures, then $(\varphi, d, \eta^{\varphi}_X)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, \coprod_{\varphi} X)$ is a morphism in $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. Hence the functor $\mathbf{T}$ acting on $(\varphi, d, \eta^{\varphi}_X)$ determines the morphism $(\mathbf{d}, \eta^{\mathbf{d}}_X)$ from $(\mathbf{\Sigma}, \mathbf{T}_{\mathbf{\Sigma}}(X))$ to $(\mathbf{\Lambda}, \mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X))$, where $\eta^{\mathbf{d}}_X = ((\eta_{\coprod_{\varphi} X})_{\varphi} \circ \eta^{\varphi}_X)^{\sharp}$ is the $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X)_{\varphi}$ that extends the S-sorted mapping $(\eta_{\coprod_{\varphi} X})_{\varphi} \circ \eta^{\varphi}_X$ from X to $\mathrm{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X)_{\varphi}$. Therefore, for every $s \in S$, $\eta^{\mathbf{d}}_{X,s}$ translates terms for $\mathbf{\Sigma}$ of type (X, s) into terms for $\mathbf{\Lambda}$ of type $(\coprod_{\varphi} X, \varphi(s))$. The $\mathbf{\Sigma}$-homomorphisms $\eta^{\mathbf{d}}_X$, as stated in the following proposition, are in fact the components of a natural transformation, and this contributes to explain their relevance as *translators*.

Proposition 2.11. Let $\mathbf{d}$ be a morphism of signatures from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$. Then the family $\eta^{\mathbf{d}} = (\eta^{\mathbf{d}}_X)_{X \in \boldsymbol{\mathcal{U}}}$, which to an S-sorted set X assigns the $\mathbf{\Sigma}$-homomorphism $\eta^{\mathbf{d}}_X$ from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X)_{\varphi}$, is a natural transformation from $\mathbf{T}_{\mathbf{\Sigma}}$ to $\mathbf{d}^* \circ \mathbf{T}_{\mathbf{\Lambda}} \circ \coprod_{\varphi}$, and so, for the forgetful functor $\mathrm{G}_{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Set}^S$, the family $\mathrm{G}_{\mathbf{\Sigma}} * \eta^{\mathbf{d}}$, i.e., the horizontal composition of the natural transformations $\eta^{\mathbf{d}}$ and $\mathrm{id}_{\mathrm{G}_{\mathbf{\Sigma}}}$, also denoted by $\eta^{\mathbf{d}}$, is a natural transformation from $\mathrm{T}_{\mathbf{\Sigma}} = \mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$ to $\Delta_{\varphi} \circ \mathrm{T}_{\mathbf{\Lambda}} \circ \coprod_{\varphi}$, taking into account that $\mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{d}^* = \Delta_{\varphi} \circ \mathrm{G}_{\mathbf{\Lambda}}$ and $\mathrm{T}_{\mathbf{\Lambda}} = \mathrm{G}_{\mathbf{\Lambda}} \circ \mathbf{T}_{\mathbf{\Lambda}}$.

The category $\mathbf{Alg}$ of algebras, as was the case for the categories $\mathbf{MSet}$ and $\mathbf{Sig}$, is also bicomplete. These results are already known, although, in particular, we are not aware of any suitably explicit and direct proof, as that provided by us below, of the cocompleteness of $\mathbf{Alg}$.

Proposition 2.12. The category $\mathbf{Alg}$ is complete.

Proof. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a signature morphism. Since the forgetful functors $\mathrm{G}_{\mathbf{\Sigma}}$ and $\mathrm{G}_{\mathbf{\Lambda}}$ create projective limits and the functors $\mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{d}^*$ and $\Delta_{\varphi} \circ \mathrm{G}_{\mathbf{\Lambda}}$ from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Set}^S$ are identical, the functor $\mathbf{d}^*$ preserves projective limits, i.e., is continuous. But the category $\mathbf{Sig}$ is complete, and, for every signature $\mathbf{\Sigma}$, $\mathbf{Alg}(\mathbf{\Sigma})$ is complete. Therefore, by Theorem 1, pp. 247–248, the category $\mathbf{Alg}$ is complete. □

To prove that the category $\mathbf{Alg}$ is cocomplete we begin by proving that, for every signature morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the functor $\mathbf{d}^*$ from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$ has a left adjoint $\mathbf{d}_*$.

Proposition 2.13. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a signature morphism. Then there exists a functor $\mathbf{d}_*$ from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Alg}(\mathbf{\Lambda})$ that is left adjoint to the functor $\mathbf{d}^*$.

Proof. We restrict ourselves to define the object and morphism mappings of $\mathbf{d}_*$. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra. Then $\mathbf{d}_*(\mathbf{A})$ is the $\mathbf{\Lambda}$-algebra defined as $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} A)/\overline{R}^{\mathbf{A}}$, where $\overline{R}^{\mathbf{A}}$ is the congruence on $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} A)$ generated by the T-sorted relation $R^{\mathbf{A}}$, defined, for every $t \in T$, as

$$R^{\mathbf{A}}_t = \left\{ \left((F^{\mathbf{A}}_{\sigma}(a_i \mid i \in |w|), s), d(\sigma)((a_i, w_i) \mid i \in |w|) \right) \;\middle|\; \begin{array}{c} s \in \varphi^{-1}[t],\ w \in S^{\star}, \\ \sigma \in \Sigma_{w,s},\ \&\ \ a \in A_w \end{array} \right\}.$$

Let f be a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{A}'$. Then $R^{\mathbf{A}} \subseteq \mathrm{Ker}(\mathrm{pr}_{\overline{R}^{\mathbf{A}'}} \circ (\coprod_{\varphi} f)^{@})$ because, for $t \in T$ and $((F_{\sigma}(a_i \mid i \in |w|), s), d(\sigma)((a_i, w_i) \mid i \in |w|)) \in R^{\mathbf{A}}_t$, we have that

$$\begin{aligned}
[(\textstyle\coprod_{\varphi} f)^{@}(F^{\mathbf{A}}_{\sigma}(a_i \mid i \in |w|), s)] &= [(f_s(F^{\mathbf{A}}_{\sigma}(a_i \mid i \in |w|)), s)] \\
&= [d(\sigma)(f_{w_i}(a_i, w_i) \mid i \in |w|)] \\
&= [(\textstyle\coprod_{\varphi} f)^{@}(d(\sigma)((a_i, w_i) \mid i \in |w|))].
\end{aligned}$$

From this it follows that there exists a unique $\mathbf{\Lambda}$-homomorphism $\mathbf{d}_*(f)$ from $\mathbf{d}_*(\mathbf{A})$ to $\mathbf{d}_*(\mathbf{A}')$ such that $\mathbf{d}_*(f) \circ \mathrm{pr}_{\overline{R}^{\mathbf{A}}} = \mathrm{pr}_{\overline{R}^{\mathbf{A}'}} \circ (\coprod_{\varphi} f)^{@}$.

We leave it to the reader to verify that the functor $\mathbf{d}_*$ thus defined is left adjoint to $\mathbf{d}^*$. □

Proposition 2.14. The category **Alg** is cocomplete.

Proof. This follows from Theorem 2, pp. 250–251, in [TBG91]. □

From Propositions 2.12 and 2.14 we obtain immediately the following

Corollary 2.15. The category **Alg** is bicomplete.

The contravariant functor Alg from **Sig** to **Cat** is not only useful to construct the category **Alg**. Actually, as we will show from here to the end of this section, Alg, together with a pseudo-functor Ter from **Sig** to **Cat**, and a pseudo-extranatural transformation (Tr, θ) (from a pseudo-functor on $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to **Cat**, induced by Alg and Ter, to the functor $\mathrm{K}_{\mathbf{C}}$, between the same categories, which picks **Set**), will enable us to construct a new institution on **Set**, the many-sorted term institution, denoted by $\mathfrak{Tm} = (\mathbf{Sig}, \mathrm{Alg}, \mathrm{Ter}, (\mathrm{Tr}, \theta))$, but for a concept of institution that is *strictly* more general than that of generalized **V**-institution defined by Goguen and Burstall in [GoB86].

For the institution $\mathfrak{Tm}$ on **Set**, as we will prove, it happens that the existence of the pseudo-functor Ter follows from the fact that, for every signature $\mathbf{\Sigma}$, the terms for $\mathbf{\Sigma}$, understood in a generalized sense to be explained below, have a categorical interpretation as the morphisms of a suitable category $\mathbf{Ter}(\mathbf{\Sigma})$. Furthermore, the component Tr of the pseudo-extranatural transformation (Tr, θ) depends for its existence on the fact that the generalized terms have canonically associated generalized term operations on the algebras. Therefore, to proceed properly, we should begin by defining, for a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ and an S-sorted set X, the concepts of many-sorted X-ary operation on $\mathbf{A}$ and of many-sorted X-ary term operation on $\mathbf{A}$, and the procedure of realization of terms P of type (X, s) as term operations $P^{\mathbf{A}}$ on $\mathbf{A}$.

Definition 2.16. Let X be an S-sorted set, $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, $s \in S$ and $P \in \mathrm{T}_{\mathbf{\Sigma}}(X)_s$ a term for $\mathbf{\Sigma}$ of type (X, s). Then the $\mathbf{\Sigma}$-algebra of the *many-sorted X-ary operations on* $\mathbf{A}$, $\mathbf{O}_X(\mathbf{A})$, is $\mathbf{A}^{A_X}$, i.e., the direct A_X-power of $\mathbf{A}$, where A_X is $\mathrm{Hom}(X, A)$, the (ordinary) set of the S-sorted mappings from X to A. For abbreviation we let *X-ary operations on* $\mathbf{A}$ stand for *many-sorted X-ary operations on* $\mathbf{A}$. The $\mathbf{\Sigma}$-algebra of the *many-sorted X-ary term operations on* $\mathbf{A}$, $\mathbf{Ter}_X(\mathbf{A})$, is the subalgebra of $\mathbf{O}_X(\mathbf{A})$ generated by $\mathcal{P}^A_X = (\mathcal{P}^A_{X,s})_{s \in S} = (\{\, \mathrm{pr}^A_{X,s,x} \mid x \in X_s \,\})_{s \in S}$, the subfamily of $\mathrm{O}_X(A) = A^{A_X}$, where, for every $s \in S$ and every $x \in X_s$, $\mathrm{pr}^A_{X,s,x}$ is the mapping from A_X to A_s which sends $a \in A_X$ to $a_s(x)$. For abbreviation we let *X-ary term operations on* $\mathbf{A}$ stand for *many-sorted X-ary term operations on* $\mathbf{A}$. We denote by $\mathrm{Tr}^{X,\mathbf{A}}$ the unique $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{O}_X(\mathbf{A})$ such that $\mathrm{pr}^A_X = \mathrm{Tr}^{X,\mathbf{A}} \circ \eta_X$, where pr^A_X is the S-sorted mapping $(\mathrm{pr}^A_{X,s})_{s \in S}$ from X to $\mathrm{O}_X(A)$ whose s-th coordinate, for each $s \in S$, is $\mathrm{pr}^A_{X,s} = (\mathrm{pr}^A_{X,s,x})_{x \in X_s}$. For abbreviation, we let $P^{\mathbf{A}}$ stand for the image of P under $\mathrm{Tr}^{X,\mathbf{A}}_s$, and we call the mapping $P^{\mathbf{A}}$ from A_X to A_s, the *term operation on* $\mathbf{A}$ *determined by* P, or the *term realization of* P *on* $\mathbf{A}$. For simplicity of notation, we continue to write $\mathrm{Tr}^{X,\mathbf{A}}$ for the co-restriction of the $\mathbf{\Sigma}$-homomorphism $\mathrm{Tr}^{X,\mathbf{A}} \colon \mathbf{T}_{\mathbf{\Sigma}}(X) \longrightarrow \mathbf{O}_X(\mathbf{A})$ to the subalgebra $\mathbf{Ter}_X(\mathbf{A})$ of $\mathbf{O}_X(\mathbf{A})$.

We note that since the above concepts are defined for arbitrary many-sorted sets, they are also applicable, in particular, for a given set of sorts S and an arbitrary, but fixed, S-sorted set of variables $V^S = (V^S_s)_{s \in S}$, where, for every $s \in S$, $V^S_s = \{\, v^s_n \mid n \in \mathbb{N} \,\}$ is a countably infinite set, to the *finite* S-sorted subsets ${\downarrow}w$ of V^S associated to the words $w \in S^{\star}$, where, for every word $w \in S^{\star}$, we let ${\downarrow}w$ stand for the finite S-sorted subset of V^S defined, for every $s \in S$, as $({\downarrow}w)_s = \{\, v^s_i \mid i \in w^{-1}[s] \,\}$.

In all that follows, every proposition relative to the above concepts will only be stated for arbitrary many-sorted sets, therefore the corresponding propositions for the finite S-sorted subsets ${\downarrow}w$ of V^S will not be actually stated and will remain tacit. However, we point out that in order to prove the just mentioned implicit propositions, it is useful to know that, for a word $w \in S^{\star}$, a mapping $\varphi\colon S \longrightarrow T$, and its extension $\varphi^{\star}\colon \mathbf{T}_{\star}(S) \longrightarrow \mathbf{T}_{\star}(T)$ to the corresponding free monoids on S and T, the S-sorted set $\downarrow w$ can be embedded in the S-sorted set $(\downarrow\varphi^{\star}(w))_{\varphi}$, associated to the T-sorted set $\downarrow\varphi^{\star}(w) \cong \coprod_{\varphi}(\downarrow w)$, through the S-sorted mapping $\mathrm{in}^{w,\varphi}$ defined, for every $s \in S$, as follows

$$\mathrm{in}^{w,\varphi}_{s} \begin{cases} (\downarrow w)_{s} & \longrightarrow \ (\downarrow\varphi^{\star}(w))_{\varphi(s)} \\ \quad v^{s}_{i} & \longmapsto \ v^{\varphi(s)}_{i} \end{cases}$$

and that this embedding has as an immediate consequence that a signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, determines a morphism $(\varphi, d, \mathrm{in}^{w,\varphi})\colon (S, \Sigma, \downarrow w) \longrightarrow (T, \Lambda, \downarrow\varphi^{\star}(w))$ in $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, hence that the s-th component of the $\mathbf{\Sigma}$-homomorphism $(\mathrm{in}^{w,\varphi})^{\mathbf{d}}\colon \mathbf{T}_{\mathbf{\Sigma}}(\downarrow w) \longrightarrow \mathbf{T}_{\mathbf{\Lambda}}(\downarrow\varphi^{\star}(w))_{\varphi}$ translates terms for $\mathbf{\Sigma}$ of type $(\downarrow w, s)$ into terms for $\mathbf{\Lambda}$ of type $(\downarrow\varphi^{\star}(w), \varphi(s))$.

For completeness we recall that for many-sorted terms, as for single-sorted terms, we also have that: (1) the *exchange law* is valid, i.e., that given a valuation $a\colon X \longrightarrow A$, where X is an S-sorted set and A the underlying many-sorted set of a $\mathbf{\Sigma}$-algebra $\mathbf{A}$, and a term P for $\mathbf{\Sigma}$ of type (X, s), we always have the equality $a^{\sharp}_{s}(P) = P^{\mathbf{A}}(a)$, and (2) that the $\mathbf{\Sigma}$-homomorphisms *commute* with term operations, i.e., that given a $\mathbf{\Sigma}$-homomorphism $u\colon \mathbf{A} \longrightarrow \mathbf{B}$ and a term P for $\mathbf{\Sigma}$ of type (X, s), we always have the equality $u_{s} \circ P^{\mathbf{A}} = P^{\mathbf{B}} \circ u_{X}$.

Following this we state the fundamental facts about term operations of different arities on the same algebra. These facts are, actually, the categorical counterpart and the generalization to the many-sorted case of some of those stated by Schmidt in [Sch61], pp. 107–109.

Proposition 2.17. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra and $f\colon X \longrightarrow Y$ an S-sorted mapping. Then there exists a unique $\mathbf{\Sigma}$-homomorphism $\mathrm{Ter}_{f}(\mathbf{A})$ from $\mathbf{Ter}_{X}(\mathbf{A})$ to $\mathbf{Ter}_{Y}(\mathbf{A})$ such that $\mathrm{Tr}^{Y,\mathbf{A}} \circ f^{@} = \mathrm{Ter}_{f}(\mathbf{A}) \circ \mathrm{Tr}^{X,\mathbf{A}}$. Besides, for every S-sorted set X, we have that $\mathrm{Ter}_{\mathrm{id}_{X}}(\mathbf{A}) = \mathrm{id}_{\mathbf{Ter}_{X}(\mathbf{A})}$, and, if $g\colon Y \longrightarrow Z$ is another S-sorted mappings, then $\mathrm{Ter}_{g \circ f}(\mathbf{A}) = \mathrm{Ter}_{g}(\mathbf{A}) \circ \mathrm{Ter}_{f}(\mathbf{A})$.

The proposition just stated can be interpreted as meaning that, for a $\mathbf{\Sigma}$-algebra $\mathbf{A}$, we have: (1) a functor $\mathrm{Ter}_{(\cdot)}(\mathbf{A})$ from $\mathbf{Set}^{S}$ to $\mathbf{Alg}(\mathbf{\Sigma})$ which sends an S-sorted set X to the $\mathbf{\Sigma}$-algebra $\mathbf{Ter}_{X}(\mathbf{A})$, and an S-sorted mapping f from X to Y to the $\mathbf{\Sigma}$-homomorphism $\mathrm{Ter}_{f}(\mathbf{A})$ from $\mathbf{Ter}_{X}(\mathbf{A})$ to $\mathbf{Ter}_{Y}(\mathbf{A})$, and (2) a natural transformation $\mathrm{Tr}^{(\cdot),\mathbf{A}}$ from $\mathbf{T}_{\mathbf{\Sigma}}$ to $\mathrm{Ter}_{(\cdot)}(\mathbf{A})$ which sends an S-sorted set X to the $\mathbf{\Sigma}$-homomorphism $\mathrm{Tr}^{X,\mathbf{A}}$ from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{Ter}_{X}(\mathbf{A})$.

What we want to prove now is the compatibility between the translation of terms and their realization as term operations on the algebras. But for this it will be shown to be useful to take into account the following auxiliary functors and natural transformation.

Definition 2.18. For a mapping $\varphi\colon S \longrightarrow T$, an S-sorted set X, a T-sorted set Y, and an S-sorted mapping $f\colon X \longrightarrow Y_{\varphi}$, we have the following functors and natural transformation. $\mathrm{H}(Y, \cdot)$ is the covariant hom-functor from $\mathbf{Set}^{T}$ to $\mathbf{Set}$. $\mathrm{H}(X, \cdot) \circ \Delta_{\varphi}$ is the functor from $\mathbf{Set}^{T}$ to $\mathbf{Set}$ which sends a T-sorted set A to the set $(A_{\varphi})_{X}$, and a T-sorted mapping u from A to B to the mapping $\mathrm{H}(X, \cdot)(u_{\varphi})$ from $(A_{\varphi})_{X}$ to $(B_{\varphi})_{X}$ which assigns to an S-sorted mapping ℓ from X to A_{φ} the mapping $u_{\varphi} \circ \ell$ from X to B_{φ}. $\vartheta^{\varphi,f}$ is the natural transformation from $\mathrm{H}(Y, \cdot)$ to $\mathrm{H}(X, \cdot) \circ \Delta_{\varphi}$ which sends a T-sorted set A to the mapping $\vartheta^{\varphi,f}_{A}$ from A_{Y} to $(A_{\varphi})_{X}$ which assigns to a morphism t in A_{Y} the morphism $t_{\varphi} \circ f$ in $(A_{\varphi})_{X}$. Therefore, for a T-sorted set A, we have the S-sorted mapping $\Upsilon^{\varphi,f}_{A}$ from $\mathrm{O}_{X}(A_{\varphi}) = A_{\varphi}^{(A_{\varphi})_{X}}$ to $\mathrm{O}_{Y}(A)_{\varphi} = (A^{A_{Y}})_{\varphi} = A_{\varphi}^{A_{Y}}$ which, for $s \in S$, sends $a\colon (A_{\varphi})_{X} \longrightarrow A_{\varphi(s)}$ to $a \circ \vartheta^{\varphi,f}_{A}\colon A_{Y} \longrightarrow A_{\varphi(s)}$.

Proposition 2.19. Let $(\varphi, d, f)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$ be a morphism in the category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. Then, for every $\mathbf{\Lambda}$-algebra $\mathbf{A}$ and term $P \in \mathrm{T}_{\mathbf{\Sigma}}(X)_{s}$ for $\mathbf{\Sigma}$ of type (X, s), the mappings $P^{\mathbf{d}^{*}(\mathbf{A})} \circ \vartheta^{\varphi,f}_{A}$ and $f^{\mathbf{d}}_{s}(P)^{\mathbf{A}}$ from A_{Y} to $A_{\varphi(s)}$ are identical.

We gather in the following corollary some useful consequences of the last proposition.

Corollary 2.20. Let $(\varphi, d, f)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$ be a morphism in the category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, $\mathbf{A}$ a

$\mathbf{\Lambda}$-algebra, and $P \in \mathrm{T}_{\mathbf{\Sigma}}(X)_s$ a term for $\mathbf{\Sigma}$ of type (X,s). Then we have that the following diagrams commute

$$\begin{tikzcd}
\mathbf{T}_{\mathbf{\Sigma}}(X) \arrow[r, "\mathrm{Tr}^{X,\mathbf{d}^*(\mathbf{A})}"] \arrow[d, "f^{\mathbf{d}}"'] & \mathbf{Ter}_X(\mathbf{d}^*(\mathbf{A})) \arrow[d, "\Upsilon^{\varphi,f}_A"] \\
\mathbf{T}_{\mathbf{\Lambda}}(Y)_\varphi \arrow[r, "\mathrm{Tr}^{Y,\mathbf{A}}_{\varphi}"'] & \mathbf{Ter}_Y(\mathbf{A})_\varphi
\end{tikzcd}
\qquad
\begin{tikzcd}
(A_\varphi)_X \arrow[r, "P^{\mathbf{d}^*(\mathbf{A})}"] & A_{\varphi(s)} \arrow[d, equal] \\
A_{\coprod_\varphi X} \arrow[u, "\theta^{\varphi}_{X,A}"] \arrow[r, "\eta^{\mathbf{d}}_{X,s}(P)^{\mathbf{A}}"'] & A_{\varphi(s)}
\end{tikzcd}$$

Proof. The left-hand diagram commutes because, for a morphism (φ,d,f) from (S,Σ,X) to (T,Λ,Y) and a $\mathbf{\Lambda}$-algebra $\mathbf{A}$, the S-sorted mapping $\Upsilon^{\varphi,f}_A$ from $\mathrm{O}_X(A_\varphi)$ to $\mathrm{O}_Y(A)_\varphi$ is actually a $\mathbf{\Sigma}$-homomorphism from $\mathbf{O}_X(\mathbf{d}^*(\mathbf{A}))$ to $\mathbf{O}_Y(\mathbf{A})_\varphi$ that restricts to $\mathbf{Ter}_X(\mathbf{d}^*(\mathbf{A}))$ and $\mathbf{Ter}_Y(\mathbf{A})_\varphi$.

The right-hand diagram commutes because, for the T-sorted set $\coprod_\varphi X$ and the S-sorted mapping η^{φ}_X from X to $(\coprod_\varphi X)_\varphi$, we have that $\vartheta^{\varphi,\eta^{\varphi}_X}_A = \theta^{\varphi}_{X,A}$. □

As it is well-known, for a signature $\mathbf{\Sigma}$, the conglomerate of terms for $\mathbf{\Sigma}$ is the set $\bigcup_{X\in\boldsymbol{\mathcal{U}}}\bigcup_{s\in S}\mathrm{T}_{\mathbf{\Sigma}}(X)_s$, but such an amorphous set is not adequate, because of its lack of structure, for some tasks, as e.g., to explain the invariant character of the realization of terms as term operations on algebras, under change of signature (or to state a Completeness Theorem for finitary many-sorted equational logic).

However, by conveniently generalizing the concept of term for a signature $\mathbf{\Sigma}$ (as explained immediately below), it is possible to endow, in a natural way, to the corresponding generalized terms for $\mathbf{\Sigma}$, taken as *morphisms*, with a category structure, that enables us to give a category-theoretic explanation of the existing relation between terms and algebras. To this we add that the use of the generalized terms and related notions, such as, e.g., that of generalized equation (to be defined in the following section), has allowed us, in [ClS05], to provide a purely category-theoretical proof of the Completeness Theorem for monads in categories of sorted sets.

Actually, we associate to every signature $\mathbf{\Sigma}$ the category $\mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})^{\mathrm{op}}$, of generalized terms for $\mathbf{\Sigma}$, that we denote, to shorten notation, by $\mathbf{Ter}(\mathbf{\Sigma})$, i.e., the dual of the Kleisli category for $\mathbb{T}_{\mathbf{\Sigma}} = (\mathrm{T}_{\mathbf{\Sigma}},\eta,\mu)$, the standard monad derived from the adjunction $\mathbf{T}_{\mathbf{\Sigma}} \dashv \mathrm{G}_{\mathbf{\Sigma}}$ between the category $\mathbf{Alg}(\mathbf{\Sigma})$ and the category $\mathbf{Set}^S$, with $\mathrm{T}_{\mathbf{\Sigma}} = \mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$.

The construction of the category $\mathbf{Ter}(\mathbf{\Sigma})$ is a natural one. This is so, essentially, because it has been obtained by applying a category-theoretic construction, concretely that of Kleisli (in [Kle65]). However, to understand more plainly how the category $\mathbf{Ter}(\mathbf{\Sigma})$ is obtained, or, more precisely, from where the morphisms of $\mathbf{Ter}(\mathbf{\Sigma})$ arise, the following observation could be helpful. For a signature $\mathbf{\Sigma}$, an S-sorted set X, and a sort $s \in S$, an ordinary term $P \in \mathrm{T}_{\Sigma}(X)_s$ for $\mathbf{\Sigma}$ of type (X,s) is, essentially, an S-sorted mapping $P\colon \delta^s \longrightarrow \mathrm{T}_{\Sigma}(X)$ where, for $s \in S$, $\delta^s = (\delta^s_t)_{t\in S}$, the delta of Kronecker in s, is the S-sorted set such that $\delta^s_t = \varnothing$ if $s \neq t$ and $\delta^s_s = 1$. But the just mentioned S-sorted mappings do not constitute the morphisms of a category. Therefore, in order to get a category, it seems natural to replace the special S-sorted sets that are the deltas of Kronecker, as domains of morphisms, by arbitrary S-sorted sets, thus obtaining the generalized terms, that are the category-theoretic rendering of the ordinary terms, since they are now S-sorted mappings from an S-sorted set to the free $\mathbf{\Sigma}$-algebra on another S-sorted set, i.e., morphisms in a category $\mathbf{Ter}(\mathbf{\Sigma})$. This category-theoretic perspective about terms, in its turn, will allow us to get a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$, of realization of terms as term operations, from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$, and therefore to define (in the next section) the validation of equations, understood as ordered pairs of coterminal terms in the corresponding generalized sense, in an algebra.

Since it will be fundamental in all that follows, we provide, for a signature $\mathbf{\Sigma}$, the full definition of the category $\mathbf{Ter}(\mathbf{\Sigma})$ and also the explicit definition of the procedure of realization of the terms for $\mathbf{\Sigma}$ as term operations on a given $\mathbf{\Sigma}$-algebra. Observe that we depart, in the definition of the category $\mathbf{Ter}(\mathbf{\Sigma})$, but only for this type of category, from the (non-Ehresmannian) tradition, in calling a category by the name of its morphisms.

Definition 2.21. Let $\mathbf{\Sigma}$ be a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra. Then $\mathbf{Ter}(\mathbf{\Sigma})$, the category of *generalized terms for* $\mathbf{\Sigma}$, is the dual of $\mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})$: the objects are the elements of $\boldsymbol{\mathcal{U}}^S$; the morphisms from an S-sorted set X to another Y, which we call *generalized terms for* $\mathbf{\Sigma}$ *of type* (X,Y), or, simply, *terms of type* (X,Y), are the S-sorted mappings P from Y to $\mathrm{T}_{\mathbf{\Sigma}}(X)$; the composition, denoted in $\mathbf{Ter}(\mathbf{\Sigma})$ and $\mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})$ by $\diamond$, is the operation which sends $P\colon X \longrightarrow Y$ and $Q\colon Y \longrightarrow Z$ in $\mathbf{Ter}(\mathbf{\Sigma})$ to $Q \diamond P\colon X \longrightarrow Z$ in $\mathbf{Ter}(\mathbf{\Sigma})$, where $Q \diamond P$ is $\mu_X \circ P^{@} \circ Q$, with μ_X the value at X of the multiplication μ of the monad $\mathbb{T}_{\mathbf{\Sigma}}$ and $P^{@}$ the value of the functor $\mathbf{T}_{\mathbf{\Sigma}}$ at the S-sorted mapping $P\colon Y \longrightarrow \mathrm{T}_{\mathbf{\Sigma}}(X)$; and the identities are the values of η, the unit of the monad $\mathbb{T}_{\mathbf{\Sigma}}$, at the S-sorted sets. If $P\colon X \longrightarrow Y$ is a term for $\mathbf{\Sigma}$ of type (X, Y), then $P^{\mathbf{A}}$, the *term operation on* $\mathbf{A}$ *determined by* P, or the *term realization of* P *on* $\mathbf{A}$, is the mapping from A_X to A_Y which assigns to a valuation f of the variables X in A the valuation $f^{\sharp} \circ P$ of the variables Y in A.

After associating to every signature $\mathbf{\Sigma}$ the corresponding category $\mathbf{Ter}(\mathbf{\Sigma})$ of terms, we proceed to assign to every signature morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ a corresponding functor $\mathbf{d}_\diamond$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})$.

Proposition 2.22. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a signature morphism. Then there exists a functor $\mathbf{d}_\diamond$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})$. Its object mapping assigns to each S-sorted set X the T-sorted set $\mathbf{d}_\diamond(X) = \coprod_\varphi X$; its morphism mapping assigns to each morphism P from X to Y in $\mathbf{Ter}(\mathbf{\Sigma})$ the morphism $\mathbf{d}_\diamond(P) = (\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_X \circ P)$ from $\coprod_\varphi X$ to $\coprod_\varphi Y$ in $\mathbf{Ter}(\mathbf{\Lambda})$, where $\eta^{\mathbf{d}}_X$ is the $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$ that extends the S-sorted mapping $(\eta_{\coprod_\varphi X})_\varphi \circ \eta^\varphi_X$ from X to $\mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$.

We state now for the generalized terms the homologous of the right-hand diagram in the first part of Corollary 2.20, i.e., the invariant character under signature change of the realization of terms as term operations in arbitrary, but fixed, algebras. We remark that from this fact we will get, in the third section, the invariance of the relation of satisfaction under signature change.

Proposition 2.23. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a signature morphism. Then, for each $\mathbf{\Lambda}$-algebra $\mathbf{A}$ and term P for $\mathbf{\Sigma}$ of type (X, Y), the mappings $P^{\mathbf{d}^*(\mathbf{A})} \circ \theta^\varphi_{X,A}$ and $\theta^\varphi_{Y,A} \circ \mathbf{d}_\diamond(P)^{\mathbf{A}}$ from $A_{\coprod_\varphi X}$ to $(A_\varphi)_Y$ are identical.

Proof. Because the S-sorted set Y is isomorphic to $\coprod_{s \in S, y \in Y_s} \delta^s$ and the functor $\coprod_\varphi$ preserves colimits, since it has Δ_φ as a right adjoint, $\coprod_\varphi Y$ is isomorphic to $\coprod_{s \in S,\, y \in Y_s} \delta^{\varphi(s)}$. But $\mathrm{Hom}(\coprod_\varphi Y, A)$ and $\prod_{s \in S, y \in Y_s} \mathrm{Hom}(\delta^{\varphi(s)}, A)$ are isomorphic, therefore it is enough to prove the proposition for the S-sorted sets of the type δ^s, i.e., the deltas of Kronecker, and this follows directly from Corollary 2.20. $\square$

Once defined the mappings that associate, respectively, to a signature the corresponding category of terms, and to a signature morphism the functor between the associated categories of terms, we state in the following proposition that both mappings are actually the components of a pseudo-functor from **Sig** to the 2-category **Cat**.

Proposition 2.24. There exists a pseudo-functor Ter from **Sig** to the 2-category **Cat** given by the following data

1. The object mapping of Ter is that which sends a signature $\mathbf{\Sigma}$ to the category $\mathrm{Ter}(\mathbf{\Sigma}) = \mathbf{Ter}(\mathbf{\Sigma})$.
2. The morphism mapping of Ter is that which sends a signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ to the functor $\mathrm{Ter}(\mathbf{d}) = \mathbf{d}_\diamond$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})$.
3. For every $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ and $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$, the natural isomorphism $\gamma^{\mathbf{d},\mathbf{e}}$ from $\mathbf{e}_\diamond \circ \mathbf{d}_\diamond$ to $(\mathbf{e} \circ \mathbf{d})_\diamond$ is that which is defined, for every S-sorted set X, as the isomorphism $\gamma^{\mathbf{d},\mathbf{e}}_X\colon \coprod_\psi \coprod_\varphi X \longrightarrow \coprod_{\psi \circ \varphi} X$ in $\mathbf{Ter}(\mathbf{\Omega})$ that corresponds to the U-sorted mapping

$$\begin{tikzcd}[column sep=large] \coprod_{\psi\circ\varphi} X \arrow[r, "(\gamma^{\varphi,\psi}_X)^{-1}"] & \coprod_\psi \coprod_\varphi X \arrow[r, "\eta_{\coprod_\psi \coprod_\varphi X}"] & \mathrm{T}_{\mathbf{\Omega}}(\coprod_\psi \coprod_\varphi X), \end{tikzcd}$$

   where $\gamma^{\varphi,\psi}_X$ is the component at X of the natural isomorphism $\gamma^{\varphi,\psi}$ for the pseudo-functor $\mathrm{MSet}^{\amalg}$.
4. For every signature $\mathbf{\Sigma}$, the natural isomorphism $\nu^{\mathbf{\Sigma}}$ from $\mathrm{Id}_{\mathbf{Ter}(\mathbf{\Sigma})}$ to $(\mathrm{id}_{\mathbf{\Sigma}})_\diamond$ is that which is defined, for every S-sorted set X, as the isomorphism $\nu^{\mathbf{\Sigma}}_X\colon X \longrightarrow \coprod_{\mathrm{id}_S} X$ in $\mathbf{Ter}(\mathbf{\Sigma})$ that corresponds to the S-sorted mapping

$$\begin{tikzcd}[column sep=large] \coprod_{\mathrm{id}_S} X \arrow[r, "\nu^S_X"] & X \arrow[r, "\eta_X"] & \mathrm{T}_{\mathbf{\Omega}}(X), \end{tikzcd}$$

where ν^S_X is the component at X of the natural isomorphism ν^S for the pseudo-functor $\mathrm{MSet}^{\amalg}$.

Our next goals are to prove that

1. For a signature $\mathbf{\Sigma}$, there is a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$, that formalizes simultaneously the procedure of realization of terms (as term operations on algebras), and its naturalness (by taking into account the variation of the algebras through the homomorphisms between them), and that
2. For a signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, there is a natural isomorphism $\theta^{\mathbf{d}}$ from $\mathrm{Tr}^{\mathbf{\Lambda}} \circ (\mathrm{Id}_{\mathbf{Alg}(\mathbf{\Lambda})} \times \mathbf{d}_\diamond)$ to $\mathrm{Tr}^{\mathbf{\Sigma}} \circ (\mathbf{d}^* \times \mathrm{Id}_{\mathbf{Ter}(\mathbf{\Sigma})})$, that shows the invariant character of the procedure of realization of terms under signature change.

To accomplish the first stated goal we begin by proving the following

Lemma 2.25. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra, P a term of type (X, Y), and Q a term of type (Y, Z). Then we have that $(Q \diamond P)^{\mathbf{A}} = Q^{\mathbf{A}} \circ P^{\mathbf{A}}$. Besides, for η_X, the identity morphism at X in $\mathbf{Ter}(\mathbf{\Sigma})$, $\eta_X^{\mathbf{A}} = \mathrm{id}_{A_X}$.

Proof. We restrict ourselves to prove the first part of the lemma because the proof of the second one is straightforward. Since $(Q \diamond P)^{\mathbf{A}}$ is the mapping from A_X to A_Z which sends an S-sorted mapping $u\colon X \longrightarrow A$ to the S-sorted mapping $u^\sharp \circ (Q \diamond P) = u^\sharp \circ \mu_X \circ P^@ \circ Q\colon Z \longrightarrow A$, (where, we recall, μ_X is the value at X of the multiplication μ of the monad $\mathbb{T}_{\mathbf{\Sigma}} = (\mathrm{T}_{\mathbf{\Sigma}}, \eta, \mu)$ and $P^@$ the value at the S-sorted mapping $P\colon Y \longrightarrow \mathrm{T}_{\mathbf{\Sigma}}(X)$ of the functor $\mathbf{T}_{\mathbf{\Sigma}}$), and $Q^{\mathbf{A}} \circ P^{\mathbf{A}}$ the mapping from A_X to A_Z which sends an S-sorted mapping $u\colon X \longrightarrow A$ to the S-sorted mapping $(u^\sharp \circ P)^\sharp \circ Q\colon Z \longrightarrow A$, to verify that $(Q \diamond P)^{\mathbf{A}} = Q^{\mathbf{A}} \circ P^{\mathbf{A}}$ it is enough to prove that the $\mathbf{\Sigma}$-homomorphisms $u^\sharp \circ \mu_X \circ P^@$ and $(u^\sharp \circ P)^\sharp$ from $\mathbf{T}_{\mathbf{\Sigma}}(Y)$ to $\mathbf{A}$ are identical. But this follows from the equation $u^\sharp \circ \mu_X \circ P^@ \circ \eta_Y = (u^\sharp \circ P)^\sharp \circ \eta_Y$, that is a consequence of the laws for the monad $\mathbb{T}_{\mathbf{\Sigma}}$ and of the equation $P^@ \circ \eta_Y = \eta_{\mathrm{T}_{\mathbf{\Sigma}}(X)} \circ P$, where η_Y is the canonical embedding of Y into $\mathrm{T}_{\mathbf{\Sigma}}(Y)$ and $\eta_{\mathrm{T}_{\mathbf{\Sigma}}(X)}$ the canonical embedding of $\mathrm{T}_{\mathbf{\Sigma}}(X)$ into $\mathrm{T}_{\mathbf{\Sigma}}(\mathrm{T}_{\mathbf{\Sigma}}(X))$. □

This lemma has as an immediate consequence the following

Corollary 2.26. Let $\mathbf{\Sigma}$ be a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra. Then there exists a functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$ which sends an S-sorted set X to the set $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}(X) = A_X$ and a term $P\colon X \longrightarrow Y$ to the mapping $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}(P) = P^{\mathbf{A}}\colon A_X \longrightarrow A_Y$, i.e., the term operation on $\mathbf{A}$ determined by P.

Therefore, from the definition of the object and morphism mappings of the functors of the type $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$, we see that they encapsulate the procedure of realization of terms. And, from the fact that they preserve identities and compositions in $\mathbf{Ter}(\mathbf{\Sigma})$, we conclude that they formally represent the two basic intuitions about the behaviour of the just named procedure, i.e., that the realization of an identity term is an identity term operation, and that the realization of a composite of two terms is the composite of their respective realizations (in the same order).

Remark 2. By identifying the $\mathbf{\Sigma}$-algebras with the $\mathbb{T}_{\mathbf{\Sigma}}$-algebras, the just stated corollary can be interpreted as meaning that every $\mathbf{\Sigma}$-algebra is a functor from $\mathbf{Ter}(\mathbf{\Sigma}) = \mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})^{\mathrm{op}}$ to $\mathbf{Set}$.

Before stating the following lemma we recall that, for an S-sorted mapping f from an S-sorted set A into another B and an S-sorted set X, f_X is the value at X of the natural transformation $\mathrm{H}(\cdot, f)$ from the contravariant functor $\mathrm{H}(\cdot, A)$ to the contravariant functor $\mathrm{H}(\cdot, B)$, both from $(\mathbf{Set}^S)^{\mathrm{op}}$ to $\mathbf{Set}$.

Lemma 2.27. Let f be a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{B}$ and P a term of type (X, Y) in $\mathbf{Ter}(\mathbf{\Sigma})$. Then the mappings $P^{\mathbf{B}} \circ f_X$ and $f_Y \circ P^{\mathbf{A}}$ from A_X to B_Y are identical, and we agree to denote it by f_P.

Proof. Given an S-sorted mapping $u\colon X \longrightarrow A$, we have that $(f \circ u)^\sharp = f \circ u^\sharp$, by the universal property of the free $\mathbf{\Sigma}$-algebra on X and taking into account that f is a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Therefore, since $P^{\mathbf{B}} \circ f_X(u) = (f \circ u)^\sharp \circ P$, and $f_Y \circ P^{\mathbf{A}}(u) = f \circ (u^\sharp \circ P)$, we have that $P^{\mathbf{B}} \circ f_X(u) = f_Y \circ P^{\mathbf{A}}(u)$. Thus $P^{\mathbf{B}} \circ f_X = f_Y \circ P^{\mathbf{A}}$. □

This lemma has as an immediate consequence the following

Corollary 2.28. Let $\mathbf{\Sigma}$ be a signature and f a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Then there exists a natural transformation $\mathrm{Tr}^{\mathbf{\Sigma},f}$ from the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$ to the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{B}}$ which sends an S-sorted set X to the mapping $\mathrm{Tr}^{\mathbf{\Sigma},f}_{X} = f_X$ from A_X to B_X. Besides, for $\mathrm{id}_{\mathbf{A}}$, the identity $\mathbf{\Sigma}$-homomorphism at $\mathbf{A}$, we have that $\mathrm{Tr}^{\mathbf{\Sigma},\mathrm{id}_{\mathbf{A}}} = \mathrm{id}_{\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}}$, and, if $g\colon \mathbf{B} \longrightarrow \mathbf{C}$ is another $\mathbf{\Sigma}$-homomorphism, then $\mathrm{Tr}^{\mathbf{\Sigma},g\circ f} = \mathrm{Tr}^{\mathbf{\Sigma},g} \circ \mathrm{Tr}^{\mathbf{\Sigma},f}$.

Therefore, the naturalness of the procedure of realization of terms as term operations on the different algebras is embodied in the natural transformations of the type $\mathrm{Tr}^{\mathbf{\Sigma},f}$.

Remark 3. By identifying the $\mathbf{\Sigma}$-homomorphisms with the $\mathbb{T}_{\mathbf{\Sigma}}$-homomorphisms, the just stated corollary can be interpreted as meaning that every $\mathbf{\Sigma}$-homomorphism f from $\mathbf{A}$ to $\mathbf{B}$ is a natural transformation from the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$ to the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{B}}$, both from $\mathbf{Ter}(\mathbf{\Sigma}) = \mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})^{\mathrm{op}}$ to $\mathbf{Set}$. In fact, each homomorphism $(\mathbf{d}, f)$ from an algebra $(\mathbf{\Sigma}, \mathbf{A})$ to another $(\mathbf{\Lambda}, \mathbf{B})$ can be identified to a morphism (in the category $(\mathbf{Cat})_{//\mathbf{Set}}$, see [Gro71], p. (sub) 186) from the object $(\mathbf{Ter}(\mathbf{\Sigma}), \mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}})$ over $\mathbf{Set}$ to the object $(\mathbf{Ter}(\mathbf{\Lambda}), \mathrm{Tr}^{\mathbf{\Lambda},\mathbf{B}})$ over $\mathbf{Set}$. Concretely, each homomorphism $(\mathbf{d}, f)\colon (\mathbf{\Sigma}, \mathbf{A}) \longrightarrow (\mathbf{\Lambda}, \mathbf{B})$ can be identified to the morphism given by the pair $(\mathbf{d}_{\diamond}, (\theta^{\varphi}_{\cdot,B})^{-1} \circ \mathrm{H}(\cdot, f))$, where $\mathrm{H}(\cdot, f)$ is the natural transformation from the contravariant hom-functor $\mathrm{H}(\cdot, A)$ to the contravariant hom-functor $\mathrm{H}(\cdot, B_{\varphi})$, and $(\theta^{\varphi}_{\cdot,B})^{-1}$ the natural isomorphism from $\mathrm{H}(\cdot, B_{\varphi})$ to $\mathrm{H}(\coprod_{\varphi}(\cdot), B)$. Let us notice that the naturalness of $(\theta^{\varphi}_{\cdot,B})^{-1} \circ \mathrm{H}(\cdot, f)$ means that, for every term P for $\mathbf{\Sigma}$ of type (X, Y), the mappings $(\theta^{\varphi}_{Y,B})^{-1} \circ \mathrm{H}(Y, f) \circ P^{\mathbf{A}}$ and $\mathbf{d}_{\diamond}(P)^{\mathbf{B}} \circ (\theta^{\varphi}_{X,B})^{-1} \circ \mathrm{H}(X, f)$, both from A_X to $B_{\coprod_{\varphi} Y}$, are identical.

From the identification of the homomorphisms between algebras in the category $\mathbf{Alg}$ to some convenient morphisms between the associated objects over $\mathbf{Set}$, we can conclude, e.g., that the concept of homomorphism as defined by Bénabou in [Bén68] (that does not allow the variation of the signature and therefore it works between algebras of the same signature (see [Bén68], p. (sub) 16, last paragraph)), corresponds, for a signature $\mathbf{\Sigma}$ and a $\mathbf{\Sigma}$-homomorphism $f\colon \mathbf{A} \longrightarrow \mathbf{B}$, to the (very special) case in which $(\mathbf{d}_{\diamond}, (\theta^{\varphi}_{\cdot,B})^{-1} \circ \mathrm{H}(\cdot, f))$ is precisely $(\mathbf{d}_{\diamond}, (\theta^{\varphi}_{\cdot,B})^{-1} \circ \mathrm{H}(\cdot, f)) = (\mathrm{Id}_{\mathbf{Ter}(\mathbf{\Sigma})}, \mathrm{H}(\cdot, f))$, i.e., definitely, it corresponds to the natural transformation $\mathrm{Tr}^{\mathbf{\Sigma},f}$ from the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$ to the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{B}}$.

For an arbitrary, but fixed, signature $\mathbf{\Sigma}$ the family of functors $(\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}})_{\mathbf{A}\in\mathbf{Alg}(\mathbf{\Sigma})}$ together with the family of natural transformations $(\mathrm{Tr}^{\mathbf{\Sigma},f})_{f\in\mathrm{Mor}(\mathbf{Alg}(\mathbf{\Sigma}))}$ are the object and morphism mappings, respectively, of a functor $\mathrm{Tr}^{\mathbf{\Sigma},(\cdot)}$ from the category $\mathbf{Alg}(\mathbf{\Sigma})$ to the exponential category $\mathbf{Set}^{\mathbf{Ter}(\mathbf{\Sigma})}$, and the functor $\mathrm{Tr}^{\mathbf{\Sigma},(\cdot)}$ will allow us to prove, in the following proposition, that there exists a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$ that formalizes the realization of terms as term operations on algebras, but taking into account the variation of the algebras through the homomorphisms between them.

Proposition 2.29. Let $\mathbf{\Sigma}$ be a signature. Then there exists a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$. Its object mapping assigns to each pair $(\mathbf{A}, X)$, formed by a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ and an S-sorted set X, the set $\mathrm{Tr}^{\mathbf{\Sigma}}(\mathbf{A}, X) = \mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}(X) = A_X$ of the S-sorted mappings from X to the underlying S-sorted set A of $\mathbf{A}$; its morphism mapping assigns to each arrow (f, P) from $(\mathbf{A}, X)$ to $(\mathbf{B}, Y)$ in $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$, the mapping $\mathrm{Tr}^{\mathbf{\Sigma}}(f, P) = f_P$ from A_X to B_Y, which is precisely $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{B}}(P) \circ \mathrm{Tr}^{\mathbf{\Sigma},f}_{X} = \mathrm{Tr}^{\mathbf{\Sigma},f}_{Y} \circ \mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}(P)$.

Proof. It follows, essentially, after Lemma 2.25. □

To accomplish the earlier stated second goal, i.e., to show the invariant character of the procedure of realization of terms under signature change, we prove, for a morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the existence of a natural isomorphism between two functors from $\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$, constructed from the functors $\mathrm{Tr}^{\mathbf{\Lambda}}$, $\mathrm{Tr}^{\mathbf{\Sigma}}$, $\mathbf{d}_{\diamond}$ and $\mathbf{d}^{*}$.

Proposition 2.30. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a signature morphism. Then the family $(\theta^{\mathbf{d}}_{\mathbf{A},X})_{(\mathbf{A},X)\in\mathbf{Alg}(\mathbf{\Lambda})\times\mathbf{Ter}(\mathbf{\Sigma})}$, written $\theta^{\mathbf{d}}$ for brevity, where, for each $(\mathbf{A}, X) \in \mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma})$, $\theta^{\mathbf{d}}_{\mathbf{A},X}$ is $\theta^{\varphi}_{X,A}$, i.e., the natural isomorphism of $\coprod_{\varphi} \dashv \Delta_{\varphi}$, is a natural isomorphism from the functor $\mathrm{Tr}^{\mathbf{\Lambda}} \circ (\mathrm{Id} \times \mathbf{d}_{\diamond})$ to the functor $\mathrm{Tr}^{\mathbf{\Sigma}} \circ (\mathbf{d}^{*} \times \mathrm{Id})$, both from the category $\mathbf{Alg}(\boldsymbol{\Lambda}) \times \mathbf{Ter}(\boldsymbol{\Sigma})$ to the category $\mathbf{Set}$, as shown in the following diagram

$$\begin{tikzcd}
\mathbf{Alg}(\boldsymbol{\Lambda}) \times \mathbf{Ter}(\boldsymbol{\Sigma}) \arrow[r, "\mathbf{d}^* \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \mathbf{d}_\diamond"'] & \mathbf{Alg}(\boldsymbol{\Sigma}) \times \mathbf{Ter}(\boldsymbol{\Sigma}) \arrow[d, "\mathrm{Tr}^{\boldsymbol{\Sigma}}"] \\
\mathbf{Alg}(\boldsymbol{\Lambda}) \times \mathbf{Ter}(\boldsymbol{\Lambda}) \arrow[r, "\mathrm{Tr}^{\boldsymbol{\Lambda}}"'] \arrow[ur, Rightarrow, shorten=12mm, "\theta^{\mathbf{d}}"] & \mathbf{Set}
\end{tikzcd}$$

Proof. Let $(f, P)\colon (\mathbf{A}, X) \longrightarrow (\mathbf{B}, Y)$ be a morphism in $\mathbf{Alg}(\boldsymbol{\Lambda}) \times \mathbf{Ter}(\boldsymbol{\Sigma})$. Then we have the following situation

$$\begin{tikzcd}
 & & (\mathbf{A}, X) \arrow[r, "(f,P)"] \arrow[dl, mapsto] & (\mathbf{B}, Y) \arrow[dr, mapsto] & & \\
(\mathbf{A}, \coprod_\varphi X) \arrow[r, "(f, \mathbf{d}_\diamond(P))"] & (\mathbf{B}, \coprod_\varphi Y) \arrow[dd, mapsto] & & (\mathbf{d}^*(\mathbf{A}), X) \arrow[r, "(f_\varphi, P)"] & (\mathbf{d}^*(\mathbf{B}), Y) \arrow[d, mapsto] & \\
 & & & & (A_\varphi)_X \arrow[ddl, "(f_\varphi)_X"'] \arrow[ddr, "P^{\mathbf{d}^*(\mathbf{A})}"] & \\
 & A_{\coprod_\varphi X} \arrow[urrr, "\theta^\varphi_{X,A}"] \arrow[ddl, "f_{\coprod_\varphi X}"'] \arrow[ddr, "\mathbf{d}_\diamond(P)^{\mathbf{A}}"] & & & & \\
 & & & (B_\varphi)_X \arrow[ddr, "P^{\mathbf{d}^*(\mathbf{B})}"] & & (A_\varphi)_Y \arrow[ddl, "(f_\varphi)_Y"] \\
B_{\coprod_\varphi X} \arrow[urrr, "\theta^\varphi_{X,B}"] \arrow[ddr, "\mathbf{d}_\diamond(P)^{\mathbf{B}}"'] & & A_{\coprod_\varphi Y} \arrow[urrr, "\theta^\varphi_{Y,A}"] \arrow[ddl, "f_{\coprod_\varphi Y}"] & & & \\
 & & & & (B_\varphi)_Y & \\
 & B_{\coprod_\varphi Y} \arrow[urrr, "\theta^\varphi_{Y,B}"'] & & & &
\end{tikzcd}$$

But the bottom diagram in the above figure commutes, because of Proposition 2.23, the naturalness of θ^φ, and the fact that f is a $\boldsymbol{\Lambda}$-homomorphism. Therefore the mappings $\theta^\varphi_{Y,B} \circ f_{\mathbf{d}_\diamond(P)}$ and $(f_\varphi)_P \circ \theta^\varphi_{X,A}$ from $A_{\coprod_\varphi X}$ to $(B_\varphi)_Y$ are identical. From this it follows that the family $\theta^{\mathbf{d}}$ is a natural isomorphism from $\mathrm{Tr}^{\boldsymbol{\Lambda}} \circ (\mathrm{Id} \times \mathbf{d}_\diamond)$ to $\mathrm{Tr}^{\boldsymbol{\Sigma}} \circ (\mathbf{d}^* \times \mathrm{Id})$. □

Our next goal is to construct the many-sorted term institution by combining adequately the above components, i.e., the contravariant functor Alg from $\mathbf{Sig}$ to $\mathbf{Cat}$, the pseudo-functor Ter from $\mathbf{Sig}$ to $\mathbf{Cat}$, the family of functors $\mathrm{Tr} = (\mathrm{Tr}^{\boldsymbol{\Sigma}})_{\boldsymbol{\Sigma} \in \mathbf{Sig}}$, and the family of natural isomorphisms $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \mathrm{Mor}(\mathbf{Sig})}$.

To attain the just stated goal we need to recall beforehand some auxiliary concepts. In particular, we proceed to define next, among others, the concept of pseudo-extranatural transformation in 2-categories and for pseudo-functors. This generality is necessary because, later on (e.g., in the fifth section), we will get a 2-category, $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$, on which we will define a pseudo-functor related by a pseudo-extranatural transformation to a definite functor (also defined on the same 2-category), from which we will get the so-called term 2-institution of Fujiwara.

Definition 2.31. Let $\mathbf{C}$ and $\mathbf{D}$ be two 2-categories, $F, G\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ two 2-functors, and (α, β) a pair such that

1. For every 0-cell c in $\mathbf{C}$, $\alpha_c\colon F(c, c) \longrightarrow G(c, c)$ is a 1-cell in $\mathbf{D}$.
2. For every 1-cell $f\colon c \longrightarrow c'$ in $\mathbf{C}$, β_f is a 2-cell in $\mathbf{D}$ from $G(1, f) \circ \alpha_c \circ F(f, 1)$ to $G(f, 1) \circ \alpha_{c'} \circ F(1, f)$.

Then we say that (α, β) is a

1. *Lax-dinatural transformation* from F to G if, for every 2-cell $\xi\colon f \Rightarrow g$ in $\mathbf{C}$, we have that
$$\beta_g \circ (G(1,\xi) * \alpha_c * F(\xi,1)) = (G(\xi,1) * \alpha_{c'} * F(1,\xi)) \circ \beta_f.$$
2. *Pseudo-dinatural transformation* from F to G if it is a lax-dinatural transformation and, for every $f\colon c \longrightarrow c'$ in $\mathbf{C}$, β_f is an isomorphism.
3. 2-*dinatural transformation* from F to G if it is a lax-dinatural transformation and, for every $f\colon c \longrightarrow c'$ in $\mathbf{C}$, β_f is an identity.

The dinatural transformations when F and G are pseudo-functors will also be relevant for us. In this case it is necessary to impose additional conditions of compatibility with the natural isomorphisms of the pseudo-functors. The definition is as follows.

Definition 2.32. Let $\mathbf{C}$ and $\mathbf{D}$ be two 2-categories, (F,γ^F,ν^F) and (G,γ^G,ν^G) two pseudo-functors from $\mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ to $\mathbf{D}$, and (α,β) a pair such that

1. For every 0-cell c in $\mathbf{C}$, $\alpha_c\colon F(c,c) \longrightarrow G(c,c)$ is a 1-cell in $\mathbf{D}$.
2. For every 1-cell $f\colon c \longrightarrow c'$ in $\mathbf{C}$, β_f is a 2-cell in $\mathbf{D}$ from $G(1,f)\circ\alpha_c\circ F(f,1)$ to $G(f,1)\circ\alpha_{c'}\circ F(1,f)$.

Then we say that (α,β) is a *lax-dinatural transformation* from (F,γ^F,ν^F) to (G,γ^G,ν^G) if it satisfies the following compatibility conditions:

1. For every 2-cell $\xi\colon f \Rightarrow g$ in $\mathbf{C}$, we have that
$$\beta_g \circ (G(1,\xi) * \alpha_c * F(\xi,1)) = (G(\xi,1) * \alpha_{c'} * F(1,\xi)) \circ \beta_f.$$
2. For every pair of 1-cells $f\colon c \longrightarrow c'$, $g\colon c' \longrightarrow c''$ in $\mathbf{C}$, we have that
$$\gamma^F_{(1,f),(1,g)} \circ \big(G(f,1) * \beta_g * F(1,f)\big) \circ \big(G(1,g) * \beta_f * F(g,1)\big) = \beta_{g\circ f} \circ \big(\gamma^G_{(1,f),(1,g)} * \alpha_c * \gamma^F_{(g,1),(f,1)}\big).$$
3. For every object c in $\mathbf{C}$, we have that
$$\alpha_c * \nu^F_{(c,c)} = \nu^G_{(c,c)} * \alpha_c.$$

If the pseudo-functor G is independent of both variables, then we say that the above transformations are *lax-extranatural*, *pseudo-extranatural* or *extranatural*, respectively. Then the compatibility with the 2-cells of $\mathbf{C}$ is equivalent to
$$\beta_g \circ (\alpha_c * F(\xi,1)) = (\alpha_{c'} * F(1,\xi)) \circ \beta_f,$$
and the compatibility of the composition of 1-cells in $\mathbf{C}$ with the natural isomorphisms of F is equivalent to
$$\gamma^F_{(1,f),(1,g)} \circ (\beta_g * F(1,f)) \circ (\beta_f * F(g,1)) = \beta_{g\circ f} \circ (\alpha_c * \gamma^F_{(g,1),(f,1)}).$$

In the following proposition we construct a pseudo-functor from the product category $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to $\mathbf{Cat}$, and prove that there exists a pseudo-extranatural transformation from it to the functor from $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to $\mathbf{Cat}$ that is constantly $\mathbf{Set}$.

But before stating and proving it we point out that subsequently it will be reformulated in a more compact form (by taking into account the directing principles of the institutional frame of Goguen and Burstall in [GoB86] and after generalizing such a frame appropriately) as asserting the existence of a certain institution on the category $\mathbf{Set}$. By doing so the conceptual and structural richness involved in the proposition will be fully and elegantly reflected in the institution structure.

Proposition 2.33. There exists a pseudo-functor $\mathrm{Alg}(\cdot) \times \mathrm{Ter}(\cdot)$ from $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to $\mathbf{Cat}$, obtained from the contravariant functor Alg and the pseudo-functor Ter, which sends a pair of signatures $(\boldsymbol{\Sigma},\boldsymbol{\Lambda})$ to the category $\mathbf{Alg}(\boldsymbol{\Sigma}) \times \mathbf{Ter}(\boldsymbol{\Lambda})$, and a pair of signature morphisms $(\mathbf{d},\mathbf{e})$ from $(\boldsymbol{\Sigma},\boldsymbol{\Lambda})$ to $(\boldsymbol{\Sigma}',\boldsymbol{\Lambda}')$ in $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to the functor $\mathbf{d}^* \times \mathbf{e}_\diamond$ from $\mathbf{Alg}(\boldsymbol{\Sigma}) \times \mathbf{Ter}(\boldsymbol{\Lambda})$ to $\mathbf{Alg}(\boldsymbol{\Sigma}') \times \mathbf{Ter}(\boldsymbol{\Lambda}')$. Furthermore, the family of functors $\mathrm{Tr} = (\mathrm{Tr}^{\boldsymbol{\Sigma}})_{\boldsymbol{\Sigma}\in\mathbf{Sig}}$, together with the family $\theta = (\theta^{\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Sig})}$, where $\theta^{\mathbf{d}}$ is the natural isomorphism of Proposition 2.30, is a pseudo-extranatural transformation from the pseudo-functor $\mathrm{Alg}(\cdot) \times \mathrm{Ter}(\cdot)$ to the functor $\mathrm{K}_{\mathbf{Set}}$, which picks $\mathbf{Set}$, both from $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to $\mathbf{Cat}$.

Proof. Because the 2-category structure of **Sig** is, in this case, trivial, we need only prove the compatibility with the natural isomorphisms of the pseudo-functor $\mathrm{Alg}(\cdot) \times \mathrm{Ter}(\cdot)$.

We restrict our attention to prove the compatibility of the composition of 1-cells in **Sig** with the natural isomorphisms of $\mathrm{Alg}(\cdot) \times \mathrm{Ter}(\cdot)$. But for this, it is enough to verify that, for every $f\colon \mathbf{A} \longrightarrow \mathbf{B}$ in $\mathbf{Alg}(\boldsymbol{\Omega})$ and $P\colon X \longrightarrow Y$ in $\mathbf{Ter}(\boldsymbol{\Sigma})$, the following diagram commutes

$$\begin{tikzcd}
 & (A_{\varphi})_{\coprod_{\varphi} X} \arrow[rr, "\theta^{\varphi}_{X,A_{\psi}}" description] \arrow[dd, "\mathbf{d}_{\diamond}(P)^{\mathbf{e}^{*}(\mathbf{A})}" description] & & (A_{\psi_{\varphi}})_{X} \arrow[dd, "P^{\mathbf{d}^{*}(\mathbf{e}^{*}(\mathbf{A}))}" description] \\
A_{\coprod_{\psi}\coprod_{\varphi} X} \arrow[ur, "\theta^{\psi}_{\coprod_{\varphi} X,A}" description] \arrow[rr, "(\gamma^{\mathbf{d},\mathbf{e}}_{X})^{\mathbf{A}}" description] \arrow[dd, "\mathbf{e}_{\diamond} \circ \mathbf{d}_{\diamond}(P)^{\mathbf{A}}" description] & & A_{\coprod_{\psi\circ\varphi} X} \arrow[ur, "\theta^{\psi\circ\varphi}_{X,A}" description] \arrow[dd, "\mathbf{e}_{\diamond}(\mathbf{d}_{\diamond}(P))^{\mathbf{A}}" description] & \\
 & (A_{\varphi})_{\coprod_{\varphi} Y} \arrow[rr, "\theta^{\varphi}_{Y,A_{\psi}}" description] \arrow[dd, "(f_{\varphi})_{\coprod_{\varphi} Y}" description] & & (A_{\psi_{\varphi}})_{Y} \arrow[dd, "(f_{\psi_{\varphi}})_{Y}" description] \\
A_{\coprod_{\psi}\coprod_{\varphi} Y} \arrow[ur, "\theta^{\psi}_{\coprod_{\varphi} Y,A}" description] \arrow[rr, "(\gamma^{\mathbf{d},\mathbf{e}}_{Y})^{\mathbf{A}}" description] \arrow[dd, "f_{\coprod_{\psi}\coprod_{\varphi} Y}" description] & & A_{\coprod_{\psi\circ\varphi} Y} \arrow[ur, "\theta^{\psi\circ\varphi}_{Y,A}" description] \arrow[dd, "f_{\coprod_{\psi\circ\varphi} Y}" description] & \\
 & (B_{\psi})_{\coprod_{\varphi} Y} \arrow[rr, "\theta^{\varphi}_{Y,B_{\psi}}" description] & & (B_{\psi_{\varphi}})_{Y} \\
B_{\coprod_{\psi}\coprod_{\varphi} Y} \arrow[ur, "\theta^{\psi}_{\coprod_{\varphi} Y,B}" description] \arrow[rr, "(\gamma^{\mathbf{d},\mathbf{e}}_{Y})^{\mathbf{B}}" description] & & B_{\coprod_{\psi\circ\varphi} Y} \arrow[ur, "\theta^{\psi\circ\varphi}_{Y,B}" description] &
\end{tikzcd}$$

And this is so in consequence of the definitions of the involved entities. □

The preceding proposition can be reformulated in a more compact form, taking into account the directing principles of the institutional frame of Goguen and Burstall in [GoB86], as asserting the existence of a certain institution on the category **Set**. By doing so the conceptual and structural richness involved in the proposition is fully and elegantly reflected in the institution structure.

To actually realize the announced reformulation of the proposition just stated we should begin by defining a concept of institution that generalizes, even more, that one defined by Goguen and Burstall in [GoB86]. This generalization is founded, ultimately, on the fact that the compatibility of generalized many-sorted terms and many-sorted algebras with respect to transformations between many-sorted signatures is also valid when generalized many-sorted terms and many-sorted algebras (of different many-sorted signatures) are endowed with natural category structures.

Definition 2.34. Let **C** be a category. Then a 2-*institution on* **C** is a quadruple $(\mathbf{Sig}, \mathrm{Mod}, \mathrm{Sen}, (\alpha, \beta))$, where

1. **Sig** is a 2-category.
2. $\mathrm{Mod}\colon \mathbf{Sig}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ a pseudo-functor.
3. $\mathrm{Sen}\colon \mathbf{Sig} \longrightarrow \mathbf{Cat}$ a pseudo-functor.
4. $(\alpha, \beta)\colon \mathrm{Mod}(\cdot) \times \mathrm{Sen}(\cdot) \longrightarrow \mathrm{K}_{\mathbf{C}}$ a pseudo-extranatural transformation.

If **Sig** is an ordinary category, instead of a 2-category, then we will speak of an *institution on* **C**.

Remark 4. The concept of 2-institution is defined relative to a category, i.e., it has meaning for a 0-cell **C** of the 2-category $\mathbf{Cat} = 1\!-\!\mathbf{Cat}$, of categories, functors, and natural transformations between functors. Therefore, if it were necessary for some application, the concept of 3-institution ought to be defined relative to a 0-cell **C** of the 3-category 2–**Cat**, of 2-categories, 2-functors, 2-natural transformations and modifications between transformations, and so forth.

Actually, 2-institutions and institutions on a category, if they are understood as pseudo-extranatural transformations, go beyond both the classical conception of semantical truth defined (mathematically for the first time, through a recursive definition of satisfaction of a formula in an arbitrary relational system by a valuation of the variables in the system) by Tarski and Vaught in [TaV57], p. 85, and the latest conception of institution in Goguen and Burstall [GoB86], p. 327.

From the above it follows, immediately, the following

Corollary 2.35. The quadruple $\mathfrak{Tm} = (\mathbf{Sig}, \mathrm{Alg}, \mathrm{Ter}, (\mathrm{Tr}, \theta))$ is an institution on the category **Set**, the so-called *many-sorted term institution*, or, to abbreviate, the *term institution*.

We close this section by pointing out that the institution $\mathfrak{Tm}$ can be qualified of basic, or fundamental, among others, by the following reasons: (1) It embodies, in a coherent way, algebras, terms, and the natural procedure of realization of terms as term operations in algebras, and (2) the many-sorted equational institution and the many-sorted specification institution (both to be defined in the third section), i.e., the core of universal algebra, are built on it.

3. Many-sorted specifications and morphisms.

In this section we begin by defining, for a signature $\mathbf{\Sigma}$, the concept of $\mathbf{\Sigma}$-equation, but for the generalized terms defined in the preceding section, the binary relation of satisfaction between $\mathbf{\Sigma}$-algebras and $\mathbf{\Sigma}$-equations, and the semantical consequence operators $\mathrm{Cn}_{\mathbf{\Sigma}}$. Then, after extending the translation of generalized terms to generalized equations, we prove the corresponding satisfaction condition, and define a pseudo-functor LEq which assigns (among others) to a signature $\mathbf{\Sigma}$, the discrete category associated to the set of all labelled $\mathbf{\Sigma}$-equations, that enables us to get the many-sorted equational institution $\mathfrak{LEq}$.

After this we define, for the generalized terms, the concepts of many-sorted specification and of many-sorted specification morphism, from which we get the corresponding category, denoted by **Spf**. Then by extending some of the notions and constructions previously developed for the category **Sig** to the category **Spf**, we get $\mathfrak{Spf}$, the many-sorted specification institution on **Set**. Besides, we prove that there exists a morphism from $\mathfrak{Spf}$ to $\mathfrak{Tm}$, the many-sorted term institution on **Set**, which, together with the canonical embedding of $\mathfrak{Tm}$ into $\mathfrak{Spf}$, makes of $\mathfrak{Tm}$ a retract of $\mathfrak{Spf}$.

We now define the equations over a given signature through the morphisms of the category of terms for the signature, what it means for an equation to be valid in an algebra, and the consequence operator on the many-sorted set of the equations.

Definition 3.1. Let $\mathbf{\Sigma}$ be a signature, X, Y two S-sorted sets and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra. Then a $\mathbf{\Sigma}$-*equation of type* (X, Y) is a pair $(P, Q)\colon X \longrightarrow Y$ of parallel morphisms in $\mathbf{Ter}(\mathbf{\Sigma})$ (hence $(P, Q) \in \mathrm{Hom}(Y, \mathrm{T}_{\mathbf{\Sigma}}(X))^2$), and a $\mathbf{\Sigma}$-*equation* is a $\mathbf{\Sigma}$-equation of type (X, Y) for some S-sorted sets X, Y. We will denote by $\mathrm{Eq}(\mathbf{\Sigma})$ the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted set of all $\mathbf{\Sigma}$-equations. A $\mathbf{\Sigma}$-equation $(P, Q)\colon X \longrightarrow Y$ is *valid* in $\mathbf{A}$, denoted by $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P, Q)$, if and only if, for every $s \in S$ and $y \in Y_s$, $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,s} (P_s(y), Q_s(y))$, i.e., $(P_s(y))^{\mathbf{A}} = (Q_s(y))^{\mathbf{A}}$. We extend this satisfaction relation between $\mathbf{\Sigma}$-algebras $\mathbf{A}$ and $\mathbf{\Sigma}$-equations $(P, Q)\colon X \longrightarrow Y$ to $\mathbf{\Sigma}$-algebras $\mathbf{A}$ and families $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$ by agreeing that $\mathbf{A} \models^{\mathbf{\Sigma}} \mathcal{E}$ if and only if, for every $X, Y \in \boldsymbol{\mathcal{U}}^S$ and $(P, Q) \in \mathcal{E}_{X,Y}$, we have that $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P, Q)$. We will denote by $\mathrm{Cn}_{\mathbf{\Sigma}}$ the endomapping of $\mathrm{Sub}(\mathrm{Eq}(\mathbf{\Sigma}))$, the set of all sub-$(\boldsymbol{\mathcal{U}}^S)^2$-sorted sets of $\mathrm{Eq}(\mathbf{\Sigma})$, which sends $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$ to $\mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E})$, where, for every $X, Y \in \boldsymbol{\mathcal{U}}^S$ and $(P, Q) \in \mathrm{Eq}(\mathbf{\Sigma})_{X,Y}$, $(P, Q) \in \mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E})_{X,Y}$ if and only if, for every $\mathbf{\Sigma}$-algebra $\mathbf{A}$, if $\mathbf{A} \models^{\mathbf{\Sigma}} \mathcal{E}$, then $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P, Q)$. We call $\mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E})$ the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted set of the *semantical consequences* of $\mathcal{E}$.

If we keep in mind that for a term $P\colon X \longrightarrow Y$ for $\mathbf{\Sigma}$ of type (X, Y), $P^{\mathbf{A}}$, the term operation on $\mathbf{A}$ determined by P, is the mapping from A_X to A_Y which assigns to an S-sorted mapping $f\colon X \longrightarrow A$ precisely $f^\sharp \circ P\colon Y \longrightarrow A$, then we get the following convenient characterization of the relation $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P, Q)$:

$$\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P, Q) \text{ iff } P^{\mathbf{A}} = Q^{\mathbf{A}}.$$

Besides, by the Completeness Theorem in [ClS05], for $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$, we have that $\mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E})$ is precisely $\mathrm{Cg}^{\Pi}_{\mathbf{Ter}(\mathbf{\Sigma})}(\mathcal{E})$, i.e., the smallest Π-compatible congruence on $\mathbf{Ter}(\mathbf{\Sigma})$ that contains $\mathcal{E}$, where the superscript Π

in the operator $\mathrm{Cg}^{\Pi}_{\mathbf{Ter}(\mathbf{\Sigma})}$ abbreviates “product”. Therefore the operator $\mathrm{Cn}_{\mathbf{\Sigma}}$ on the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted set $\mathrm{Eq}(\mathbf{\Sigma})$ is a closure operator.

Remark 5. It is true that, for a signature $\mathbf{\Sigma}$, in order to equationally characterize the varieties (resp., the finitary varieties) of $\mathbf{\Sigma}$-algebras it is enough to consider the S-finite (resp., the finite) subsets of an arbitrary, but fixed, S-sorted set V^S with a countable infinity of variables in each coordinate. However, the generalized terms and equations proposed in this paper, besides containing as particular cases the ordinary terms and equations, respectively, have proved their worth, e.g., in the proof of the Completeness Theorem for monads in categories of sorted sets in [ClS05], and can also be used to attain a truly category-theoretic understanding of the subject matter (through the theory of monads).

Remark 6. The concept of equational deduction can be explained, from the standpoint of category theory, as a pseudo-functor. Actually, it is enough to define: (1) the category $\mathbf{MClSp}$, of many-sorted closure spaces, (2), for a Grothendieck universe $\boldsymbol{\mathcal{V}}$ such that $\boldsymbol{\mathcal{U}} \in \boldsymbol{\mathcal{V}}$, the 2-category $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$ of monads (i.e., pairs $(\mathbf{C}, \mathbb{T})$ such that $\mathbf{C}$, the underlying category of the monad, is in $\boldsymbol{\mathcal{V}}$ and $\mathbb{T}$ is a monad in $\mathbf{C}$), algebraic morphisms between monads (which are adjoint squares satisfying a compatibility condition), and transformations between algebraic morphisms (which are a special type of adjoint square satisfying an additional condition), into which the category $\mathbf{MClSp}$ is naturally embedded, and (3) to prove the existence of a pseudo-functor Cn from $\mathbf{Sig}$ to $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$ that has as components, essentially, the consequence operators $\mathrm{Cn}_{\mathbf{\Sigma}}$ for the different signatures $\mathbf{\Sigma}$.

By recalling that every signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ determines a functor $\mathbf{d}_\diamond$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})$, and taking into account the above definition of the equations for a signature, we next formalize the procedure of translation, by means of a signature morphism, of equations for a signature into equations for another signature in the following definition.

Definition 3.2. Let $\mathbf{d}$ be a signature morphism from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$. Then $\mathbf{d}$ induces a many-sorted mapping $((\coprod_\varphi)^2, \mathbf{d}_\diamond^2)\colon ((\boldsymbol{\mathcal{U}}^S)^2, \mathrm{Eq}(\mathbf{\Sigma})) \longrightarrow ((\boldsymbol{\mathcal{U}}^T)^2, \mathrm{Eq}(\mathbf{\Lambda}))$, the so-called *translation of equations for* $\mathbf{\Sigma}$ *into equations for* $\mathbf{\Lambda}$ *relative to* $\mathbf{d}$, where $(\coprod_\varphi)^2$ is the mapping from $(\boldsymbol{\mathcal{U}}^S)^2$ to $(\boldsymbol{\mathcal{U}}^T)^2$ which sends a pair of S-sorted sets (X, Y) to the pair $(\coprod_\varphi X, \coprod_\varphi Y)$ of T-sorted sets, and $\mathbf{d}_\diamond^2$ the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted mapping which to a $\mathbf{\Sigma}$-equation (P, Q) of type (X, Y) assigns the $\mathbf{\Lambda}$-equation $(\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q))$ of type $(\coprod_\varphi X, \coprod_\varphi Y)$.

Once defined the translation of equations, we prove in the following lemma the invariance of the relation of satisfaction under signature change, also known, for those following the terminology coined by Goguen and Burstall in [GoB84], p. 229, as the *satisfaction condition.*

Lemma 3.3. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a signature morphism, (P, Q) a $\mathbf{\Sigma}$-equation of type (X, Y) and $\mathbf{A}$ a $\mathbf{\Lambda}$-algebra. Then we have that $\mathbf{d}^*(\mathbf{A}) \models^{\mathbf{\Sigma}}_{X,Y} (P, Q)$ iff $\mathbf{A} \models^{\mathbf{\Lambda}}_{\coprod_\varphi X, \coprod_\varphi Y} (\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q))$.

Proof. The condition $\mathbf{d}^*(\mathbf{A}) \models^{\mathbf{\Sigma}}_{X,Y} (P, Q)$ is equivalent to $P^{\mathbf{d}^*(\mathbf{A})} = Q^{\mathbf{d}^*(\mathbf{A})}$. Moreover, by Proposition 2.23, $P^{\mathbf{d}^*(\mathbf{A})} = Q^{\mathbf{d}^*(\mathbf{A})}$, is equivalent to $\mathbf{d}_\diamond(P)^{\mathbf{A}} = \mathbf{d}_\diamond(Q)^{\mathbf{A}}$. Therefore $\mathbf{d}^*(\mathbf{A}) \models^{\mathbf{\Sigma}}_{X,Y} (P, Q)$ is equivalent to the condition $\mathbf{A} \models^{\mathbf{\Lambda}}_{\coprod_\varphi X, \coprod_\varphi Y} (\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q))$. □

Related to the quasi-triviality of the (short and conceptual) proof of Lemma 3.3 (as a consequence, essentially, of the fact that it is, ultimately, rooted in Proposition 2.23), perhaps it would be convenient to recall that Goguen and Burstall, in [GoB84], p. 228, have omitted the corresponding proof because they qualify it as being not entirely trivial.

To construct the many-sorted equational institution we now define a pseudo-functor LEq on the category of signatures. In order to do so we need to assume, besides the Grothendieck universe $\boldsymbol{\mathcal{U}}$, the existence of another one $\boldsymbol{\mathcal{V}}$ such that $\boldsymbol{\mathcal{U}} \in \boldsymbol{\mathcal{V}}$. The new Grothendieck universe $\boldsymbol{\mathcal{V}}$ will be used to construct the appropriate target 2-categories. Therefore, to exclude any misunderstanding, we agree to denote those categories $\mathbf{C}$ properly depending on $\boldsymbol{\mathcal{V}}$ by $\mathbf{C}_{\boldsymbol{\mathcal{V}}}$. However, since the additional assumption of a universe $\boldsymbol{\mathcal{V}}$ such that $\boldsymbol{\mathcal{U}} \in \boldsymbol{\mathcal{V}}$, will be used, almost, exclusively in this section, we do not label those categories depending on $\boldsymbol{\mathcal{U}}$ with the subscript $\boldsymbol{\mathcal{U}}$, such as has been done until now.

Definition 3.4. We denote by LEq the pseudo-functor from $\mathbf{Sig}$ to $\mathbf{Cat}_{\boldsymbol{\mathcal{V}}}$ given by the following data

1. The object mapping of LEq is that which sends a signature $\boldsymbol{\Sigma}$ to the discrete category $\mathbf{LEq}(\boldsymbol{\Sigma})$ canonically associated to the set $\bigcup_{X,Y\in\boldsymbol{\mathcal{U}}}(\mathrm{Hom}(Y,\mathrm{T}_{\boldsymbol{\Sigma}}(X))^2\times\{(X,Y)\})$ of *labelled* $\boldsymbol{\Sigma}$*-equations*, i.e., the set of all pairs $((P,Q),(X,Y))$ with (P,Q) a $\boldsymbol{\Sigma}$-equation of type (X,Y), for some $X,Y\in\boldsymbol{\mathcal{U}}$.
2. The morphism mapping of LEq is that which sends a signature morphism $\mathbf{d}$ from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$ to the functor $\mathrm{LEq}(\mathbf{d})$ from $\mathbf{LEq}(\boldsymbol{\Sigma})$ to $\mathbf{LEq}(\boldsymbol{\Lambda})$ which assigns to the labelled equation $((P,Q),(X,Y))$ in $\mathbf{LEq}(\boldsymbol{\Sigma})$ the labelled equation $\mathrm{LEq}(\mathbf{d})((P,Q),(X,Y))=((\mathbf{d}_\diamond(P),\mathbf{d}_\diamond(Q)),(\coprod_\varphi X,\coprod_\varphi Y))$ in $\mathbf{LEq}(\boldsymbol{\Lambda})$.

Corollary 3.5. The quadruple $\mathfrak{LEq}=(\mathbf{Sig},\mathrm{Alg},\mathrm{LEq},(\models,\theta))$ is an institution on $\mathbf{2}$, the so-called *many-sorted equational institution*, or, to abbreviate, the *equational institution.*

Following this we proceed to define the concept of many-sorted specification and that of many-sorted specification morphism.

Definition 3.6. A *many-sorted specification* is a pair $(\boldsymbol{\Sigma},\mathcal{E})$, where $\boldsymbol{\Sigma}$ is a signature while $\mathcal{E}\subseteq\mathrm{Eq}(\boldsymbol{\Sigma})$. A *many-sorted specification morphism* from $(\boldsymbol{\Sigma},\mathcal{E})$ to $(\boldsymbol{\Lambda},\mathcal{H})$ is a signature morphism $\mathbf{d}\colon\boldsymbol{\Sigma}\longrightarrow\boldsymbol{\Lambda}$ such that $\mathbf{d}^2_\diamond[\mathcal{E}]\subseteq\mathrm{Cn}_{\boldsymbol{\Lambda}}(\mathcal{H})$. Henceforth, to shorten terminology, we will say *specification* and *specification morphism* instead of *many-sorted specification* and *many-sorted specification morphism*, respectively. Besides, if in a specification $(\boldsymbol{\Sigma},\mathcal{E})$ the set $\mathcal{E}$ of equations is closed, i.e., $\mathrm{Cn}_{\boldsymbol{\Sigma}}(\mathcal{E})=\mathcal{E}$, then we call $(\boldsymbol{\Sigma},\mathcal{E})$ a *theory.* To abbreviate, we write, sometimes, $\overline{\mathcal{E}}$ instead of $\mathrm{Cn}_{\boldsymbol{\Sigma}}(\mathcal{E})$.

Proposition 3.7. The specifications and the specification morphisms determine a category denoted as $\mathbf{Spf}$.

Proof. We restrict ourselves to prove that the composition of specification morphisms is a specification morphism. But before proving this let us notice that if $\mathbf{d}\colon\boldsymbol{\Sigma}\longrightarrow\boldsymbol{\Lambda}$ and $\mathbf{e}\colon\boldsymbol{\Lambda}\longrightarrow\boldsymbol{\Omega}$ are signature morphisms, (P,Q) a $\boldsymbol{\Sigma}$-equation of type (X,Y), and $\mathbf{C}$ a $\boldsymbol{\Omega}$-algebra, then $\mathbf{e}_\diamond(\mathbf{d}_\diamond(P))^{\mathbf{C}}=\mathbf{e}_\diamond(\mathbf{d}_\diamond(Q))^{\mathbf{C}}$ if and only if $(\mathbf{e}\circ\mathbf{d})_\diamond(P)^{\mathbf{C}}=(\mathbf{e}\circ\mathbf{d})_\diamond(Q)^{\mathbf{C}}$. Thus, for each family of $\boldsymbol{\Sigma}$-equations $\mathcal{E}$, we have that $\mathrm{Cn}_{\boldsymbol{\Omega}}(\mathbf{e}^2_\diamond[\mathbf{d}^2_\diamond[\mathcal{E}]])=\mathrm{Cn}_{\boldsymbol{\Omega}}((\mathbf{e}\circ\mathbf{d})^2_\diamond[\mathcal{E}])$. Now, if $\mathbf{d}\colon(\boldsymbol{\Sigma},\mathcal{E})\longrightarrow(\boldsymbol{\Lambda},\mathcal{H})$ and $\mathbf{e}\colon(\boldsymbol{\Lambda},\mathcal{H})\longrightarrow(\boldsymbol{\Omega},\mathcal{F})$ are specification morphisms, then $\mathbf{e}^2_\diamond[\mathbf{d}^2_\diamond[\mathcal{E}]]\subseteq\mathbf{e}^2_\diamond[\mathrm{Cn}_{\boldsymbol{\Lambda}}(\mathcal{H})]\subseteq\mathrm{Cn}_{\boldsymbol{\Omega}}(\mathbf{e}^2_\diamond[\mathcal{H}])\subseteq\mathrm{Cn}_{\boldsymbol{\Omega}}(\mathcal{F})$, from which the proposition follows. □

Remark 7. The category $\mathbf{Th}_{\mathrm{b}}$ with objects the theories and morphisms from one theory to another the, so-called by Bénabou in [Bén68], p. (sub) 27, *banal* morphisms (also known as *axiom-preserving* morphisms), is $\mathbf{Th}_{\mathrm{b}}=\int^{\mathbf{Sig}}\mathrm{Fix}\circ\mathrm{Cn}$, where Fix is the contravariant functor from $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$ to $\mathbf{Cat}_{\boldsymbol{\mathcal{V}}}$ which sends a monad $(\mathbf{C},\mathbb{T})$ for $\boldsymbol{\mathcal{V}}$, to the preordered set $\mathbf{Fix}(\mathbb{T})=(\mathrm{Fix}(\mathbb{T}),\preccurlyeq)$, of the fixed points of $\mathbb{T}$, being $\mathrm{Fix}(\mathbb{T})$ the set of all $\mathbb{T}$-algebras (A,δ) such that the structural morphism δ from $\mathrm{T}(A)$ to A is an isomorphism, and $\preccurlyeq$ the preorder on $\mathrm{Fix}(\mathbb{T})$ defined by imposing that $(A,\delta)\preccurlyeq(A',\delta')$ iff there exists a $\mathbb{T}$-homomorphism from (A,δ) to (A',δ'). Therefore, informally speaking, we can say that the world of theories, $\mathbf{Th}_{\mathrm{b}}$, is the totalization over $\mathbf{Sig}$ of the fixed points of the consequences.

We state next some, obvious, relations between the categories $\mathbf{Sig}$ and $\mathbf{Spf}$. Every signature $\boldsymbol{\Sigma}$ determines the specification $(\boldsymbol{\Sigma},\varnothing)$, the so-called *indiscrete specification*, from which we get an inclusion functor $\mathrm{sp}_{\mathrm{i}}\colon\mathbf{Sig}\longrightarrow\mathbf{Spf}$ that is a left adjoint to the forgetful functor $\mathrm{sig}\colon\mathbf{Spf}\longrightarrow\mathbf{Sig}$ which sends an specification $(\boldsymbol{\Sigma},\mathcal{E})$ to the underlying signature $\boldsymbol{\Sigma}$. Besides, $\mathbf{Sig}$ is a retract of $\mathbf{Spf}$, i.e., $\mathrm{sig}\circ\mathrm{sp}_{\mathrm{i}}=\mathrm{Id}_{\mathbf{Sig}}$. The functor sig also has a right adjoint $\mathrm{sp}_{\mathrm{d}}\colon\mathbf{Sig}\longrightarrow\mathbf{Spf}$ which sends a signature $\boldsymbol{\Sigma}$ to $(\boldsymbol{\Sigma},\mathrm{Eq}(\boldsymbol{\Sigma}))$, the so-called *discrete specification.*

What we want now is to lift the contravariant functor Alg from $\mathbf{Sig}$ to $\mathbf{Cat}$ to the category $\mathbf{Spf}$.

Proposition 3.8. There exists a contravariant functor $\mathrm{Alg}^{\mathrm{sp}}$ from $\mathbf{Spf}$ to $\mathbf{Cat}$. Its object mapping assigns to each specification $(\boldsymbol{\Sigma},\mathcal{E})$ the category $\mathrm{Alg}^{\mathrm{sp}}(\boldsymbol{\Sigma},\mathcal{E})=\mathbf{Alg}(\boldsymbol{\Sigma},\mathcal{E})$ of its models, i.e., the full subcategory of $\mathbf{Alg}(\boldsymbol{\Sigma})$ determined by those $\boldsymbol{\Sigma}$-algebras which satisfy all the equations in $\mathcal{E}$; its morphism mapping assigns to each specification morphism $\mathbf{d}$ from $(\boldsymbol{\Sigma},\mathcal{E})$ to $(\boldsymbol{\Lambda},\mathcal{H})$ the functor $\mathrm{Alg}^{\mathrm{sp}}(\mathbf{d})=\mathbf{d}^*$ from $\mathbf{Alg}(\boldsymbol{\Lambda},\mathcal{H})$ to $\mathbf{Alg}(\boldsymbol{\Sigma},\mathcal{E})$, obtained from the functor $\mathbf{d}^*$ from $\mathbf{Alg}(\boldsymbol{\Lambda})$ to $\mathbf{Alg}(\boldsymbol{\Sigma})$ by bi-restriction.

Proof. Let $\mathbf{B}$ be a $\boldsymbol{\Lambda}$-algebra such that $\mathbf{B}\models^{\boldsymbol{\Lambda}}\mathcal{H}$. Then $\mathbf{B}\models^{\boldsymbol{\Lambda}}\mathrm{Cn}_{\boldsymbol{\Lambda}}(\mathcal{H})$, therefore $\mathbf{B}\models^{\boldsymbol{\Lambda}}\mathbf{d}^2_\diamond[\mathcal{E}]$ hence, by Lemma 3.3, $\mathbf{d}^*(\mathbf{B})\models^{\boldsymbol{\Sigma}}\mathcal{E}$. □

By applying the EG-construction to $\mathrm{Alg}^{\mathrm{sp}}$ we get the category $\int^{\mathbf{Spf}}\mathrm{Alg}^{\mathrm{sp}}$, denoted by $\mathbf{Alg}^{\mathrm{sp}}$. The category $\mathbf{Alg}^{\mathrm{sp}}$ has as objects the pairs $((\boldsymbol{\Sigma},\mathcal{E}),\mathbf{A})$, where $(\boldsymbol{\Sigma},\mathcal{E})$ is a specification and $\mathbf{A}$ a $\boldsymbol{\Sigma}$-algebra which is a model of $\mathcal{E}$, and as morphisms from $((\boldsymbol{\Sigma},\mathcal{E}),\mathbf{A})$ to $((\boldsymbol{\Lambda},\mathcal{H}),\mathbf{B})$, the pairs $(\mathbf{d},f)$, with $\mathbf{d}$ a specification morphism from $(\boldsymbol{\Sigma},\mathcal{E})$ to $(\boldsymbol{\Lambda},\mathcal{H})$ and f a $\boldsymbol{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^*(\mathbf{B})$.

Remark 8. The category **Alg** is embedded into $\mathbf{Alg}^{\mathrm{sp}}$ as a retract (essentially, because **Sig** is a retract of **Spf**).

On the other hand, taking care of the Completeness Theorem in [ClS05], every family of equations $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$ determines a congruence on the category $\mathbf{Ter}(\mathbf{\Sigma})$, hence a quotient category $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$. Besides, this procedure can be completed, as stated in the following proposition, to a pseudo-functor $\mathrm{Ter}^{\mathrm{sp}}$ from **Spf** to **Cat**, and the restriction of $\mathrm{Ter}^{\mathrm{sp}}$ to **Sig** is precisely the pseudo-functor Ter.

Proposition 3.9. There exists a pseudo-functor $\mathrm{Ter}^{\mathrm{sp}}$ from **Spf** to **Cat** defined as follows

1. $\mathrm{Ter}^{\mathrm{sp}}$ sends a specification $(\mathbf{\Sigma},\mathcal{E})$ to the category $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{\Sigma},\mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma},\mathcal{E})$, where $\mathbf{Ter}(\mathbf{\Sigma},\mathcal{E})$ is the quotient category $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$.
2. $\mathrm{Ter}^{\mathrm{sp}}$ sends a specification morphism $\mathbf{d}$ from $(\mathbf{\Sigma},\mathcal{E})$ to $(\mathbf{\Lambda},\mathcal{H})$ to the functor $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d})$, also occasionally denoted by $\mathbf{d}_\diamond$, from $\mathbf{Ter}(\mathbf{\Sigma},\mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$ to $\mathbf{Ter}(\mathbf{\Lambda},\mathcal{H}) = \mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$, which assigns to a morphism $[P]_{\overline{\mathcal{E}}}\colon X \longrightarrow Y$ in $\mathbf{Ter}(\mathbf{\Sigma},\mathcal{E})$ the morphism $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d})([P]_{\overline{\mathcal{E}}}) = [\mathbf{d}_\diamond(P)]_{\overline{\mathcal{H}}}\colon \coprod_\varphi X \longrightarrow \coprod_\varphi Y$ in $\mathbf{Ter}(\mathbf{\Lambda},\mathcal{H})$.

Proof. Everything follows, essentially, from the fact that the action of $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d})$ on $[P]_{\overline{\mathcal{E}}}$ is well defined since $\mathcal{E} \subseteq \mathrm{Ker}(\mathrm{Pr}_{\overline{\mathcal{H}}} \circ \mathbf{d}_\diamond)$, where $\mathrm{Pr}_{\overline{\mathcal{H}}}$ is the projection from $\mathbf{Ter}(\mathbf{\Lambda})$ to $\mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$. □

After this we prove that the family of functors $\mathrm{Tr} = (\mathrm{Tr}^{\mathbf{\Sigma}})_{\mathbf{\Sigma}\in\mathbf{Sig}}$, defined in Proposition 2.29, can be lifted to the family of functors $\mathrm{Tr}^{\mathrm{sp}} = (\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Sigma},\mathcal{E})})_{(\mathbf{\Sigma},\mathcal{E})\in\mathbf{Spf}}$.

Proposition 3.10. Let $(\mathbf{\Sigma},\mathcal{E})$ be a specification. Then from the product category $\mathbf{Alg}(\mathbf{\Sigma},\mathcal{E}) \times \mathbf{Ter}(\mathbf{\Sigma},\mathcal{E})$ to the category **Set** there exists a functor $\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Sigma},\mathcal{E})}$. Its object mapping assigns to each pair $(\mathbf{A},X)$, formed by a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ which satisfies $\mathcal{E}$ and an S-sorted set X, the set $\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Sigma},\mathcal{E})}(\mathbf{A},X) = A_X$ of the S-sorted mappings from X to the underlying S-sorted set A of $\mathbf{A}$; its morphism mapping assigns to each arrow $(f,[P]_{\overline{\mathcal{E}}})$ from $(\mathbf{A},X)$ to $(\mathbf{B},Y)$ the mapping $\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Sigma},\mathcal{E})}(f,[P]_{\overline{\mathcal{E}}}) = f_P$ from A_X to B_Y.

Proof. Everything follows from the fact that the action of $\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Sigma},\mathcal{E})}$ on $(f,[P]_{\overline{\mathcal{E}}})$ is well defined because from $[P]_{\overline{\mathcal{E}}} = [Q]_{\overline{\mathcal{E}}}$ it follows that, for every $\mathbf{\Sigma}$-algebra $\mathbf{C}$ which satisfies $\mathcal{E}$, $P^{\mathbf{C}} = Q^{\mathbf{C}}$. □

Next we state that the family of natural isomorphisms $\theta = (\theta^{\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Sig})}$, defined in Proposition 2.30, can be lifted to the family of natural isomorphisms $\theta^{\mathrm{sp}} = (\theta^{\mathrm{sp},\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Spf})}$.

Proposition 3.11. Let $\mathbf{d}$ be a specification morphism from $(\mathbf{\Sigma},\mathcal{E})$ to $(\mathbf{\Lambda},\mathcal{H})$. Then there exists a natural isomorphism $\theta^{\mathrm{sp},\mathbf{d}} = (\theta^{\mathrm{sp},\mathbf{d}}_{\mathbf{A},X})_{(\mathbf{A},X)\in\mathbf{Alg}(\mathbf{\Lambda},\mathcal{H})\times\mathbf{Ter}(\mathbf{\Sigma},\mathcal{E})}$ as shown in the following diagram

$$\begin{tikzcd}
\mathbf{Alg}(\mathbf{\Lambda},\mathcal{H}) \times \mathbf{Ter}(\mathbf{\Sigma},\mathcal{E}) \arrow[r, "\mathbf{d}^* \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \mathbf{d}_\diamond"'] & \mathbf{Alg}(\mathbf{\Sigma},\mathcal{E}) \times \mathbf{Ter}(\mathbf{\Sigma},\mathcal{E}) \arrow[d, "\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Sigma},\mathcal{E})}"] \arrow[dl, phantom, "\overset{\theta^{\mathrm{sp},\mathbf{d}}}{\Rightarrow}"] \\
\mathbf{Alg}(\mathbf{\Lambda},\mathcal{H}) \times \mathbf{Ter}(\mathbf{\Lambda},\mathcal{H}) \arrow[r, "\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Lambda},\mathcal{H})}"'] & \mathbf{Set}
\end{tikzcd}$$

where, for every $(\mathbf{A},X) \in \mathbf{Alg}(\mathbf{\Lambda},\mathcal{H}) \times \mathbf{Ter}(\mathbf{\Sigma},\mathcal{E})$, $\theta^{\mathrm{sp},\mathbf{d}}_{\mathbf{A},X}$ is $\theta^{\varphi}_{X,A}$, i.e., the value at (X,A) of the natural isomorphism of the adjunction $\coprod_\varphi \dashv \Delta_\varphi$.

From these two last propositions it follows immediately the following

Corollary 3.12. The quadruple $\mathfrak{Spf} = (\mathbf{Spf}, \mathrm{Alg}^{\mathrm{sp}}, \mathrm{Ter}^{\mathrm{sp}}, (\mathrm{Tr}^{\mathrm{sp}}, \theta^{\mathrm{sp}}))$ is an institution on the category **Set**, the so-called *many-sorted specification institution*, or, to abbreviate, the *specification institution.*

On the other hand, from the contravariant functor $\mathrm{Alg}^{\mathrm{sp}}$, from **Spf** to **Cat**, to the contravariant functor $\mathrm{Alg} \circ \mathrm{sig}^{\mathrm{op}}$, between the same categories, there exists a natural transformation, In, which sends a specification $(\mathbf{\Sigma},\mathcal{E})$ to the full embedding $\mathrm{In}_{(\mathbf{\Sigma},\mathcal{E})}$ of $\mathbf{Alg}(\mathbf{\Sigma},\mathcal{E})$ into $\mathbf{Alg}(\mathbf{\Sigma})$. Besides, from the pseudo-functor $\mathrm{Ter} \circ \mathrm{sig}$, from **Spf** to **Cat**, to the pseudo-functor $\mathrm{Ter}^{\mathrm{sp}}$, between the same categories, there exists a (strict) pseudo-natural transformation, Pr, given by the following data

1. For each specification $(\mathbf{\Sigma}, \mathcal{E})$, the projection functor $\mathrm{Pr}_{\overline{\mathcal{E}}}$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to the quotient category $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$.
2. For each specification morphism $\mathbf{d}$ from $(\mathbf{\Sigma}, \mathcal{E})$ to $(\mathbf{\Lambda}, \mathcal{H})$, the identity natural transformation, denoted in this case by $\mathrm{Pr}_{\mathbf{d}}$, from the functor $\mathrm{Pr}_{\overline{\mathcal{H}}} \circ (\mathrm{Ter} \circ \mathrm{sig})(\mathbf{d})$ to the functor $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d}) \circ \mathrm{Pr}_{\overline{\mathcal{E}}}$, both from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$.

Therefore we have obtained the following

Corollary 3.13. The pair $(\mathrm{sig}, (\mathrm{In}, \mathrm{Pr}))$ is a morphism of institutions from the many-sorted specification institution $\mathfrak{Spf}$ to the many-sorted term institution $\mathfrak{Tm}$.

Remark 9. Since, obviously, $\mathfrak{Tm}$ is embedded in $\mathfrak{Spf}$, taking into account the just stated corollary, we can assert that $\mathfrak{Tm}$ is, in fact, a retract of $\mathfrak{Spf}$.

4. Morphisms of Fujiwara and the term institution of Fujiwara.

In Mathematics it is standard to compare pairs of objects by means of homomorphisms, i.e., mappings from one of them to the other which relate, in a predetermined way, the primitive operations on the source object to the corresponding primitive operations on the target object. But there are natural examples of comparisons between objects, e.g., the derivations in ring theory (see [Jac68], pp. 169–172), which can only be stated by relating the primitive operations on the source object to corresponding (*families of*) *derived* operations on the target object, thus showing the necessity of conveniently generalizing the ordinary concept of homomorphism.

In this section, following the work by Fujiwara in [Fuj59], we generalize the morphisms in **Sig** in such a way that a signature morphism from a signature into another, to be called henceforth a *morphism of Fujiwara*, or more briefly, a *polyderivor*, will consist of two suitably related mappings: On the one hand, a mapping which relates the sets of sorts of the signatures and assigns to each sort in the first, a derived sort in the second, i.e., a word on the set of sorts of the second, and, on the other hand, a mapping which assigns to each formal operation in the first, a family of terms in the second, all in such a way that both transformations are compatible. This type of signature morphism, from which we will get a category $\mathbf{Sig}_{\mathfrak{pd}}$, with the same objects that **Sig**, will allow us to generalize, concordantly, the morphisms between algebras.

We will also state that the category $\mathbf{Sig}_{\mathfrak{pd}}$ of signatures and polyderivors is isomorphic to the Kleisli category for a monad in **Sig**, and that fact will confirm, to some extend, the naturalness of the concept of polyderivor. Furthermore, the contravariant functor $\mathrm{Alg}\colon \mathbf{Sig} \longrightarrow \mathbf{Cat}$ will be lifted to a contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}\colon \mathbf{Sig}_{\mathfrak{pd}} \longrightarrow \mathbf{Cat}$ and, by applying, once more, the EG-construction, we will get the category $\mathbf{Alg}_{\mathfrak{pd}}$ of algebras and algebra morphisms that will have the polyderivors as a component. In the same way, the pseudo-functor $\mathrm{Ter}\colon \mathbf{Sig} \longrightarrow \mathbf{Cat}$ will be lifted to a pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}\colon \mathbf{Sig}_{\mathfrak{pd}} \longrightarrow \mathbf{Cat}$.

On the other hand, to account exactly for the invariant character of the realization of terms in algebras under the polyderivors, we prove the existence of a pseudo-extranatural transformation (Tr, θ) from a pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}_{\mathfrak{pd}}(\cdot)$, on $\mathbf{Sig}_{\mathfrak{pd}}^{\mathrm{op}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$, to the functor $\mathrm{K}_{\mathbf{Set}}$, between the same categories, which picks **Set**.

Since it will be used afterwards, we define next the Hall and Bénabou algebras. However, before defining the aforementioned algebras (as the models of suitable specifications), we agree that for a set of sorts U, a word $x \in U^{\star}$ and a standard U-sorted set of variables $V^{U} = (\{\, v_{n}^{u} \mid n \in \mathbb{N} \,\})_{u \in U}$, $\downarrow x$ is the U-sorted subset of V^{U} defined, for every $u \in U$ as $(\downarrow x)_{u} = \{\, v_{i}^{u} \mid i \in x^{-1}[u] \,\}$, this will apply, in particular, when $U = S^{\star} \times S$ or $U = S^{\star} \times S^{\star}$.

Definition 4.1. Let S be a set of sorts and V^{H_S} the $S^{\star} \times S$-sorted set of variables $(V_{w,s})_{(w,s) \in S^{\star} \times S}$ where, for every $(w,s) \in S^{\star} \times S$, $V_{w,s} = \{\, v_{n}^{w,s} \mid n \in \mathbb{N} \,\}$. A *Hall algebra for* S is a $\mathrm{H}_S = (S^{\star} \times S, \Sigma^{\mathrm{H}_S}, \mathcal{E}^{\mathrm{H}_S})$-algebra, where Σ^{H_S} is the $S^{\star} \times S$-sorted signature, i.e., the $(S^{\star} \times S)^{\star} \times (S^{\star} \times S)$-sorted set, defined as follows:

HS_1. For every $w \in S^{\star}$ and $i \in |w|$, $\pi_{i}^{w}\colon \lambda \longrightarrow (w, w_i)$, where $|w|$ is the *length* of the word w and λ the *empty word* in $(S^{\star} \times S)^{\star}$.

HS_2. For every u, $w \in S^{\star}$ and $s \in S$, the formal operation of *generalized composition*:

$$\xi_{u,w,s}\colon ((w,s), (u,w_0), \ldots, (u, w_{|w|-1})) \longrightarrow (u,s);$$

while $\mathcal{E}^{\mathrm{H}_S}$ is the many-sorted subset of $\mathrm{Eq}(\Sigma^{\mathrm{H}_S})$ defined as follows:

H_1. *Projection.* For every u, $w \in S^\star$ and $i \in |w|$, the equation:

$$\xi_{u,w,w_i}(\pi^w_i, v^{u,w_0}_0, \ldots, v^{u,w_{|w|-1}}_{|w|-1}) = v^{u,w_i}_i$$

of type $(((u, w_0), \ldots, (u, w_{|w|-1})), (u, w_i))$.

H_2. *Identity.* For every $u \in S^\star$ and $j \in |u|$, the equation:

$$\xi_{u,u,u_j}(v^{u,u_j}_j, \pi^u_0, \ldots, \pi^u_{|u|-1}) = v^{u,u_j}_j$$

of type $(((u, u_j)), (u, u_j))$.

H_3. *Associativity.* For every u, v, $w \in S^\star$ and $s \in S$, the equation:

$$\begin{aligned}\xi_{u,v,s}(\xi_{v,w,s}(v^{w,s}_0, v^{v,w_0}_1, \ldots, v^{v,w_{|w|-1}}_{|w|}), v^{u,v_0}_{|w|+1}, \ldots, v^{u,v_{|v|-1}}_{|w|+|v|}) = {}&\\ \xi_{u,w,s}(v^{w,s}_0, \xi_{u,v,w_0}(v^{v,w_0}_1, v^{u,v_0}_{|w|+1}, \ldots, v^{u,v_{|v|-1}}_{|w|+|v|}), \ldots,&\\ \xi_{u,v,w_{|w|-1}}(v^{v,w_{|w|-1}}_{|w|}, v^{u,v_0}_{|w|+1}, \ldots, v^{u,v_{|v|-1}}_{|w|+|v|}))&\end{aligned}$$

of type $(((w, s), (v, w_0), \ldots, (v, w_{|w|-1}), (u, v_0), \ldots, (u, v_{|v|-1})), (u, s))$.

We denote by $\mathbf{Alg}(\mathrm{H}_S)$ the category of Hall algebras for S and homomorphisms between Hall algebras. Since $\mathbf{Alg}(\mathrm{H}_S)$ is a variety, the forgetful functor $\mathrm{G}_{\mathrm{H}_S}$ from $\mathbf{Alg}(\mathrm{H}_S)$ to $\mathbf{Set}^{S^\star \times S}$ has a left adjoint $\mathbf{T}_{\mathrm{H}_S}$ which assigns to an $S^\star \times S$-sorted set Σ the corresponding free Hall algebra $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$.

For every S-sorted signature Σ, $\mathrm{HT}_S(\Sigma) = (\mathrm{T}_\Sigma(\downarrow w)_s)_{(w,s) \in S^\star \times S}$ is endowed with a Hall algebra structure that formalizes the concept of substitution. Let $\mathbf{HT}_S(\Sigma)$ denote the corresponding Σ^{H_S}-algebra, then $\mathbf{HT}_S(\Sigma)$ is isomorphic to $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$. We remark that the existence of this isomorphism is interesting because it enables us, on the one hand, to get a more tractable description of the terms in $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$, and, on the other hand, to give an alternative, but equivalent, definition of the concept of derivor (defined by Goguen, Thatcher and Wagner in [GTW76], p. 86) between signatures.

Definition 4.2. Let S be a set of sorts and V^{B_S} the $(S^\star)^2$-sorted set of variables $(V_{u,w})_{(u,w) \in (S^\star)^2}$ where, for every $(u, w) \in (S^\star)^2$, $V_{u,w} = \{\, v^{u,w}_n \mid n \in \mathbb{N} \,\}$. A *Bénabou algebra for S* is a $\mathrm{B}_S = ((S^\star)^2, \Sigma^{\mathrm{B}_S}, \mathcal{E}^{\mathrm{B}_S})$-algebra, where Σ^{B_S} is the $(S^\star)^2$-sorted signature defined as follows:

BS_1. For every $w \in S^\star$ and $i \in |w|$, the formal operation of *projection*: $\pi^w_i \colon \lambda \longrightarrow (w, (w_i))$, where λ is the empty word in $S^\star$.

BS_2. For every u, $w \in S^\star$, the formal operation of *finite tupling*:

$$\langle\,\rangle_{u,w} \colon ((u, (w_0)), \ldots, (u, (w_{|w|-1}))) \longrightarrow (u, w).$$

BS_3. For every u, x, $w \in S^\star$, the formal operation of *substitution*:

$$\circ_{u,x,w} \colon ((u, x), (x, w)) \longrightarrow (u, w);$$

while $\mathcal{E}^{\mathrm{B}_S}$ is the many-sorted subset of $\mathrm{Eq}(\Sigma^{\mathrm{B}_S})$ defined as follows:

B_1. For every u, $w \in S^\star$ and $i \in |w|$, the equation:

$$\pi^w_i \circ_{u,w,(w_i)} \langle v^{u,(w_0)}_0, \ldots, v^{u,(w_{|w|-1})}_{|w|-1} \rangle_{u,w} = v^{u,(w_i)}_i,$$

of type $(((u, (w_0)), \ldots, (u, (w_{|w|-1}))), (u, (w_i)))$.

B_2. For every u, $w \in S^\star$, the equation:

$$v^{u,w}_0 \circ_{u,u,w} \langle \pi^u_0, \ldots, \pi^u_{|u|-1} \rangle_{u,u} = v^{u,w}_0,$$

of type $(((u, w)), (u, w))$.

B_3. For every u, $w \in S^\star$, the equation:

$$\langle \pi^w_0 \circ_{u,w,w_0} v^{u,w}_0, \ldots, \pi^w_{|w|-1} \circ_{u,w,w_{|w|-1}} v^{u,w}_0 \rangle_{u,w} = v^{u,w}_0,$$

of type $(((u, w)), (u, w))$.

B$_4$. For every $w \in S^\star$, the equation:

$$\langle \pi^w_0 \rangle_{w,(w_0)} = \pi^w_0,$$

of type $(((w,(w_0))),(w,(w_0)))$.

B$_5$. For every u, x, w, $y \in S^\star$, the equation:

$$v^{w,y}_0 \circ_{u,w,y} (v^{x,w}_1 \circ_{u,x,w} v^{u,x}_2) = (v^{w,y}_0 \circ_{x,w,y} v^{x,w}_1) \circ_{u,x,y} v^{u,x}_2,$$

of type $(((w,y),(x,w),(u,x)),(u,y))$,

where $v^{u,w}_n$ is the n-th variable of type (u,w), $Q \circ_{u,x,w} P$ stands for $\circ_{u,x,w}(P,Q)$, and, for abbreviation, we let $\langle P_0,\ldots,P_{|w|-1}\rangle_{u,w}$ stand for $\langle\,\rangle_{u,w}(P_0,\ldots,P_{|w|-1})$.

Since $\mathbf{Alg}(\mathrm{B}_S)$ is a variety, the forgetful functor $\mathrm{G}_{\mathrm{B}_S}$ from $\mathbf{Alg}(\mathrm{B}_S)$ to $\mathbf{Set}^{S^\star\times S^\star}$ has a left adjoint $\mathbf{T}_{\mathrm{B}_S}$ which assigns to an $S^\star \times S^\star$-sorted set the corresponding free Bénabou algebra.

For every S-sorted signature Σ, $\mathrm{BT}_S(\Sigma) = (\mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow w)_u)_{(w,u)\in S^\star\times S^\star}$, that is naturally isomorphic to $(\mathrm{Hom}(\downarrow u, \mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow w))_{(w,u)\in S^\star\times S^\star}$, is endowed with a Bénabou algebra structure. Let $\mathbf{BT}_S(\Sigma)$ denote the corresponding Σ^{B_S}-algebra, then $\mathbf{BT}_S(\Sigma)$ is isomorphic to the free Bénabou algebra $\mathbf{T}_{\mathrm{B}_S}(\coprod_{1\times\between_S}\Sigma)$ on $\coprod_{1\times\between_S}\Sigma$. We remark that the existence of this isomorphism enables us, on the one hand, to get a more tractable description of the terms in $\mathbf{T}_{\mathrm{B}_S}(\coprod_{1\times\between_S}\Sigma)$, and, on the other hand, to give an alternative, but equivalent, definition of the concept of polyderivor between signatures.

Next we define the concept of polyderivor, and we warn the reader about the convenience of looking the notational conventions stated in the introduction that have to do with the notion of monoid.

Definition 4.3. Let $\boldsymbol{\Sigma}$ and $\boldsymbol{\Lambda}$ be signatures. A *polyderivor from* $\boldsymbol{\Sigma}$ *to* $\boldsymbol{\Lambda}$ is a pair $\mathbf{d} = (\varphi, d)$, where $\varphi\colon S \longrightarrow T^\star$ while $d\colon \Sigma \longrightarrow \mathrm{BT}_T(\Lambda)_{\varphi^\sharp\times\varphi}$. Therefore, if $\mathbf{d}\colon \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ is a polyderivor, then, for every $(w,s) \in S^\star \times S$, we have that

$$d_{w,s}\colon \Sigma_{w,s} \longrightarrow \mathrm{BT}_T(\Lambda)_{\varphi^\sharp(w),\varphi(s)} (= \mathrm{T}_{\boldsymbol{\Lambda}}(\downarrow\varphi^\sharp(w))_{\varphi(s)}),$$

and, since $\Delta_{\varphi^\sharp\times\varphi} = \Delta_{1\times\between_S} \circ \Delta_{\varphi^\sharp\times\varphi^\sharp}$ and the functor $\coprod_{1\times\between_S}$ is left adjoint to the functor $\Delta_{1\times\between_S}$, d is, essentially, an $S^\star \times S^\star$-sorted mapping

$$\theta^{1\times\between_S}(d)\colon \textstyle\coprod_{1\times\between_S}\Sigma \longrightarrow \mathrm{BT}_T(\Lambda)_{\varphi^\sharp\times\varphi^\sharp}.$$

Henceforth, for every polyderivor $\mathbf{d}$, we identify d and $\theta^{1\times\between_S}(d)$.

For every signature $\boldsymbol{\Lambda}$, $\mathrm{BT}_T(\Lambda)$ is the underlying many-sorted set of $\mathbf{BT}_T(\Lambda)$, the Bénabou algebra for (T,Λ), and $\mathbf{BT}_T(\Lambda)$ is isomorphic to $\mathbf{T}_{\mathrm{B}_T}(\coprod_{1\times\between_T}\Lambda)$. Hence the polyderivors can also be defined as pairs $\mathbf{d} = (\varphi, d)$, where $\varphi\colon S \longrightarrow T^\star$ while d is an $S^\star \times S$-sorted mapping from Σ to $\mathrm{T}_{\mathrm{B}_T}(\coprod_{1\times\between_T}\Lambda)_{\varphi^\sharp\times\varphi}$, or, equivalently, an $S^\star \times S^\star$-sorted mapping from $\coprod_{1\times\between_S}\Sigma$ to $\mathrm{T}_{\mathrm{B}_T}(\coprod_{1\times\between_T}\Lambda)_{\varphi^\sharp\times\varphi^\sharp}$.

Example 1. Let $\boldsymbol{\Sigma}$ be a signature and $p \in \mathbb{N}$. Then taking

1. As $\varphi\colon S \longrightarrow S^\star$ the mapping which sends $s \in S$ to the word $\curlywedge_{\mu\in p}(s)$ and,
2. For $(w,s) \in S^\star \times S$, as $d_{w,s}$ the mapping from $\Sigma_{w,s}$ to $\mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow\varphi^\sharp(w))^p_s$ (since, in this case, $\mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow\varphi^\sharp(w))_{\varphi(s)} = \mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow\varphi^\sharp(w))^p_s$), which sends $\sigma \in \Sigma_{w,s}$ to $(\sigma(v^{w_0}_0, v^{w_1}_p, \ldots, v^{w_{|w|-1}}_{(|w|-1)p}), \ldots, \sigma(v^{w_0}_{p-1}, v^{w_1}_{2p-1}, \ldots, v^{w_{|w|-1}}_{|w|p-1}))$, in $\mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow\varphi^\sharp(w))^p_s$,

we have that $\mathbf{d} = (\varphi, d)$ is a polyderivor from $\boldsymbol{\Sigma}$ into itself.

We notice that there are , clear and natural, polyderivors between the (many-sorted) signatures of Hall and Bénabou.

Example 2. Let $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ and $\Lambda = (\Lambda_n)_{n\in\mathbb{N}}$ be two single-sorted signatures and let (Φ, P) be, with $\Phi = \{\, \varphi_\mu \mid \mu \in p \,\}$, a family of basic mapping-formulas from Σ to Λ as defined by Fujiwara in [Fuj59], p. 155. Then by associating

1. To the single-sorted signatures Σ and Λ, the signatures $(1, (\Sigma_{n,0})_{(n,0)\in 1^\star\times 1})$ and $(1, (\Lambda_{n,0})_{(n,0)\in 1^\star\times 1})$, respectively, where, for every $n \in 1^\star \cong \mathbb{N}$, $\Sigma_{n,0} = \Sigma_n$ and $\Lambda_{n,0} = \Lambda_n$, and

2. To the morphism (Φ, P) the pair (κ_p, d), where κ_p is the mapping from 1 to $1^\star$ which sends $0 \in 1$ to $p \in 1^\star$ and d the $1^\star \times 1$-sorted mapping from $(\Sigma_{n,0})_{(n,0)\in 1^\star\times 1}$ to $(\mathrm{T}_{\mathbf{\Lambda}}(\downarrow\kappa_p^\sharp(n))_{\kappa_p(0)})_{(n,0)\in 1^\star\times 1} \cong (\mathrm{T}_{\mathbf{\Lambda}}(\Phi \times \downarrow v_n)^p)_{n\in\mathbb{N}}$ which, for $n \in 1^\star$, sends $\sigma \in \Sigma_n$ to $d_{n,0}(\sigma) = (P^n_{\varphi_0,\sigma}, \ldots, P^n_{\varphi_{p-1},\sigma})$,

we have that the families of basic mapping-formulas defined by Fujiwara for the single-sorted case fall, obviously, under the concept of polyderivor. Consequently, all the examples provided by Fujiwara in [Fuj59], pp. 155–156, once reformulated as just said, are also examples of polyderivors.

Example 3. Let (φ, d) be a standard signature morphism from a signature (S, Σ) into another (T, Λ). Then from $\varphi\colon S \longrightarrow T$ we get $\between_T \circ\varphi\colon S \longrightarrow T^\star$, and from $d\colon \Sigma \longrightarrow \Lambda_{\varphi^\star\times\varphi}$, because there exists a canonical embedding from $\Lambda_{\varphi^\star\times\varphi}$ into $(\coprod_{1\times\between_T}\Lambda)_{(\between_T\circ\varphi)^\sharp\times(\between_T\circ\varphi)}$, we get the composite mapping

$$\Sigma \xrightarrow{d} \Lambda_{\varphi^\star\times\varphi} \longrightarrow (\textstyle\coprod_{1\times\between_T}\Lambda)_{(\between_T\circ\varphi)^\sharp\times(\between_T\circ\varphi)} \longrightarrow \mathrm{T}_{\mathrm{B}_T}(\textstyle\coprod_{1\times\between_T}\Lambda)_{(\between_T\circ\varphi)^\sharp\times(\between_T\circ\varphi)}.$$

Thus the standard signature morphisms fall under the concept of polyderivor.

Our next goal is to define the composition of polyderivors in order to get the category $\mathbf{Sig}_{\mathfrak{pd}}$, of signatures and polyderivors. To attain the just stated goal it is necessary to know the concept of derivor from a signature into another as defined by Goguen, Thatcher and Wagner in [GTW76], p. 86, as well as to set out also some of its properties.

Definition 4.4. Let $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ be signatures. Then a *derivor from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$ is a pair $\mathbf{d} = (\varphi, d)$, with $\varphi\colon S \longrightarrow T$ and $d\colon \Sigma \longrightarrow \mathrm{HT}_T(\Lambda)_{\varphi^\star\times\varphi}$. Therefore, if $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ is a derivor, then, for every $(w, s) \in S^\star \times S$, we have that

$$d_{w,s}\colon \Sigma_{w,s} \longrightarrow \mathrm{HT}_T(\Lambda)_{\varphi^\star(w),\varphi(s)} (= \mathrm{T}_\Lambda(\downarrow\varphi^\star(w))_{\varphi(s)})$$

sends a formal operation $\sigma\colon w \longrightarrow s$ to a term $d_{w,s}(\sigma)\colon \varphi^\star(w) \longrightarrow \varphi(s)$, and all in such a way that the arities and coarities are preserved, modulus the correspondence between the sorts given by the mapping φ.

For a signature $\mathbf{\Lambda}$ we have that $\mathrm{HT}_T(\Lambda)$ is the underlying many-sorted set of $\mathbf{HT}_T(\Lambda)$, the Hall algebra for (T, Λ). But $\mathbf{HT}_T(\Lambda)$ is isomorphic to $\mathbf{T}_{\mathrm{H}_T}(\Lambda)$, the free H_T-algebra on Λ. Consequently the derivors can be defined, alternative, but equivalently, as pairs $\mathbf{d} = (\varphi, d)$ with $\varphi\colon S \longrightarrow T$ and $d\colon \Sigma \longrightarrow \mathrm{T}_{\mathrm{H}_T}(\Lambda)$. Thus, taking into account the equivalence between the categories $\mathbf{Alg}(\mathrm{H}_T)$ and $\mathbf{Alg}(\mathrm{B}_T)$, we can state the following

Corollary 4.5. Every derivor is a polyderivor (although, obviously, not every polyderivor is a derivor).

An example of derivor originating from the field of sentential logic is that provided by Gödel (see [Göd33]) in his work about an interpretation of the intuitionistic propositional logic into a modal extension of the classical propositional logic.

Example 4. Let $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ be a single-sorted signature such that $\Sigma_1 = \{\neg_{\mathrm{i}}\}$, $\Sigma_2 = \{\wedge_{\mathrm{i}}, \vee_{\mathrm{i}}, \rightarrow_{\mathrm{i}}\}$ and $\Sigma_n = \varnothing$, if $n \neq 1, 2$, $\Lambda = (\Lambda_n)_{n\in\mathbb{N}}$ a single-sorted signature such that $\Lambda_1 = \{\neg_{\mathrm{c}}, \Box\}$, $\Lambda_2 = \{\wedge_{\mathrm{c}}, \vee_{\mathrm{c}}, \rightarrow_{\mathrm{c}}\}$, and $\Lambda_n = \varnothing$, if $n \neq 1, 2$, and $g = (g_n)_{n\in\mathbb{N}}$ the family defined, for $n \neq 1, 2$, as the unique mapping from $\varnothing$ to $\mathrm{T}_\Lambda(\downarrow v_n)$, and, for $n = 1, 2$, as follows

1. $g_1(\neg_{\mathrm{i}}) = \neg_{\mathrm{c}}\Box v_0$.
2. $g_2(\wedge_{\mathrm{i}}) = v_0 \wedge_{\mathrm{c}} v_1$.
3. $g_2(\vee_{\mathrm{i}}) = \Box v_0 \vee_{\mathrm{c}} \Box v_1$.
4. $g_2(\rightarrow_{\mathrm{i}}) = \Box v_0 \rightarrow_{\mathrm{c}} \Box v_1$.

Then d is a derivor from Σ to Λ. This derivor defines the intuitionistic connectives in terms of the classical connectives together with $\Box$, the operator of necessity.

Next we proceed to define the composition of derivors and to state that the corresponding category of signatures and derivors, denoted by $\mathbf{Sig}_{\mathfrak{d}}$, is isomorphic to the Kleisli category for a monad $\mathbb{T}_{\mathfrak{d}}$ in $\mathbf{Sig}$. This last result means, in other words, that the derivors are indiscernible from the morphisms of the Kleisli category for the monad $\mathbb{T}_{\mathfrak{d}}$ in $\mathbf{Sig}$, thus confirming its mathematical naturalness. By proceeding in this way we, on the one hand, move one step forward, from the standpoint of category theory, in the investigation of some of the most notable positive properties of the category $\mathbf{Sig}_{\mathfrak{d}}$, with regard to what has been done in [GTW76], and, on the other hand, get a model on which to base the subsequent work that we have to do concerning polyderivors.

We point out that the definition of the composition of derivors, in strong contrast with that of polyderivors below, is based on the standard specification morphisms between Hall specifications. Actually, if instead of starting from a mapping $\varphi\colon S \longrightarrow T^\star$, as is the case for the polyderivors, we start from an ordinary mapping $\varphi\colon S \longrightarrow T$, then, as we state next, we get a functor $(\varphi^\star \times \varphi, h^\varphi)^*$ from the category $\mathbf{Alg}(\mathrm{H}_T)$, of Hall algebras for T, to the category $\mathbf{Alg}(\mathrm{H}_S)$, of Hall algebras for S (and the existence of such a functor will follow from that of a specification morphism from $(S^\star \times S, \Sigma^{\mathrm{H}_S}, \mathcal{E}^{\mathrm{H}_S})$ to $(T^\star \times T, \Sigma^{\mathrm{H}_T}, \mathcal{E}^{\mathrm{H}_T})$). This functor, in its turn, will allows us to endow to the many-sorted set $\mathrm{HT}_T(\Lambda)_{\varphi^\star\times\varphi}$ with a Hall algebra structure for S, from which the composition of derivors will be defined explicitly.

Proposition 4.6. Let $\varphi\colon S \longrightarrow T$ be a mapping. Then the $S^\star \times S$-sorted mapping $h^\varphi\colon \Sigma^{\mathrm{H}_S} \longrightarrow \Sigma^{\mathrm{H}_T}_{\varphi^\star\times\varphi}$ defined as follows

1. For every $w \in S^\star$ and $i \in |w|$, $h^\varphi(\pi^w_i) = \pi^{\varphi^\star(w)}_i$,
2. For every u, $w \in S^\star$ and $s \in S$, $h^\varphi(\xi_{u,w,s}) = \xi_{\varphi^\star(u),\varphi^\star(w),\varphi(s)}$,

is such that $(\varphi^\star \times \varphi, h^\varphi)\colon (S^\star \times S, \Sigma^{\mathrm{H}_S}, \mathcal{E}^{\mathrm{H}_S}) \longrightarrow (T^\star \times T, \Sigma^{\mathrm{H}_T}, \mathcal{E}^{\mathrm{H}_T})$ is a specification morphism. Therefore $\varphi\colon S \longrightarrow T$ induces a functor $(\varphi^\star \times \varphi, h^\varphi)^*$ from $\mathbf{Alg}(\mathrm{H}_T)$ to $\mathbf{Alg}(\mathrm{H}_S)$ which sends $\mathbf{HT}_T(\Lambda)$, the free Hall algebra on a T-sorted signature Λ, to a Hall algebra for S, with $\mathrm{HT}_T(\Lambda)_{\varphi^\star\times\varphi}$ as underlying $S^\star \times S$-sorted set.

For a derivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the many-sorted mapping d from Σ to $\mathrm{HT}_T(\Lambda)_{\varphi^\star\times\varphi}$ can be lifted to a homomorphism of Hall algebras $d^\sharp$ from $\mathbf{HT}_S(\Sigma)$ to $\mathbf{HT}_T(\Lambda)_{\varphi^\star\times\varphi}$, whose underlying $S^\star \times S$-sorted mapping determines a translation of terms for $\mathbf{\Sigma}$ to terms for $\mathbf{\Lambda}$. In particular, for every $(w,s) \in S^\star \times S$, $d^\sharp_{w,s}$ assigns to terms in $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow w)_s$ terms in $\mathrm{T}_{\mathbf{\Lambda}}(\downarrow \varphi^\sharp(w))_{\varphi(s)}$.

Before we define immediately below the composition of derivors and the identities we recall that $\mathbf{\Sigma}$, $\mathbf{\Lambda}$, $\mathbf{\Omega}$, and $\mathbf{\Xi}$ denote the signatures (S,Σ), (T,Λ), (U,Ω), and (X,Ξ), respectively, and $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{h}$ denote the derivors (φ, d), (ψ, e), and (γ, h), respectively.

Definition 4.7. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ and $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$ be derivors. Then $\mathbf{e} \circ \mathbf{d}$, the *composition of* $\mathbf{d}$ *and* $\mathbf{e}$, is the derivor $(\psi \circ \varphi, e^\sharp_{\varphi^\star\times\varphi} \circ d)$, where $e^\sharp_{\varphi^\star\times\varphi} \circ d$ is obtained from

$$\begin{tikzcd}
\Lambda \arrow[r, "\eta^{\mathrm{H}_T}_{\Lambda}"] \arrow[dr, "e"'] & \mathrm{HT}_T(\Lambda) \arrow[d, "e^\sharp"] & & \mathrm{HT}_T(\Lambda)_{\varphi^\star\times\varphi} \arrow[d, "e^\sharp_{\varphi^\star\times\varphi}"] & \Sigma \arrow[l, "d"'] \\
& \mathrm{HT}_U(\Omega)_{\psi^\star\times\psi} & \text{as} & \mathrm{HT}_U(\Omega)_{\psi^\star\times\psi_{\varphi^\star\times\varphi}} &
\end{tikzcd}$$

where $e^\sharp$ is the extension of e to the free Hall algebra on Λ. On the other hand, for every signature $\mathbf{\Sigma}$, the *identity at* $\mathbf{\Sigma}$ is $(\mathrm{id}_S, \eta^{\mathrm{H}_S}_{\Sigma})$.

The preceding definition allows us to get a corresponding, and explicit, category of signatures and derivors.

Proposition 4.8. The signatures together with the derivors determine a category, that we denote by $\mathbf{Sig}_{\mathfrak{d}}$.

We point out that the category $\mathbf{Sig}_{\mathfrak{d}}$ of signatures and derivors can be obtained, in a natural way, as an isomorphic copy of the Kleisli category for a monad in $\mathbf{Sig}$. This is founded on the fact that, for every set of sorts S, we have the adjunction $\mathbf{T}_{\mathrm{H}_S} \dashv \mathrm{G}_{\mathrm{H}_S}$, from which we get the monad $\mathbb{T}_{\mathrm{H}_S} = (\mathrm{T}_{\mathrm{H}_S}, \eta^{\mathrm{H}_S}, \mu^{\mathrm{H}_S})$ in $\mathbf{Set}^{S^\star\times S}$, that canonically induces the monad $\mathbb{T}_{\mathfrak{d}}$ in $\mathbf{Sig}$ at issue.

Proposition 4.9. The triple $\mathbb{T}_{\mathfrak{d}} = (\mathfrak{d}, \eta^{\mathfrak{d}}, \mu^{\mathfrak{d}})$, where

1. $\mathfrak{d}$ is the functor which sends a signature $\mathbf{\Sigma}$ to the signature $(S, \mathrm{T}_{\mathrm{H}_S}(\Sigma))$, and a signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ to the signature morphism $(\varphi, d^\sharp)$ from $(S, \mathrm{T}_{\mathrm{H}_S}(\Sigma))$ to $(T, \mathrm{T}_{\mathrm{H}_T}(\Lambda))$,
2. $\eta^{\mathfrak{d}}_{\mathbf{\Sigma}} = (\mathrm{id}_S, \eta^{\mathrm{H}_S}_{\Sigma})$, with $\eta^{\mathrm{H}_S}_{\Sigma}$ the value at Σ of the unit η^{H_S} of the monad $\mathbb{T}_{\mathrm{H}_S}$, and

3. $\mu^{\mathfrak{d}}_{\boldsymbol{\Sigma}} = (\mathrm{id}_S, \mu^{\mathrm{H}_S}_{\Sigma})$, with $\mu^{\mathrm{H}_S}_{\Sigma}$ the value at Σ of the multiplication μ^{H_S} of the monad $\mathbb{T}_{\mathrm{H}_S}$,

is a monad in **Sig** and the categories $\mathbf{Sig}_{\mathfrak{d}}$ and $\mathbf{Kl}(\mathbb{T}_{\mathfrak{d}})$ are isomorphic.

Remark 10. Almost all the results about the categories **Sig**, **Alg**, and **Spf** established in the second and third section, suitably extended, are also valid for the corresponding categories $\mathbf{Sig}_{\mathfrak{d}}$, $\mathbf{Alg}_{\mathfrak{d}}$, and $\mathbf{Spf}_{\mathfrak{d}}$. But the derivors being a particular case of the polyderivors, we restrict ourselves to unfold those results only for the polyderivors.

Corollary 4.10. The category $\mathbf{Sig}_{\mathfrak{d}}$ has coproducts.

Proof. This follows from Proposition 3.5, pp. 122–123, in [ClS05]. □

Our next goal is to define the composition of two polyderivors. To attain it we begin by stating that every mapping φ from S to $T^{\star}$ determines a functor $(\varphi^{\sharp} \times \varphi^{\sharp}, b^{\varphi})^{*}$ from $\mathbf{Alg}(\mathrm{B}_T)$ to $\mathbf{Alg}(\mathrm{B}_S)$ (observe that such a functor is induced not by a standard specification morphism between Bénabou specifications, but by a *derivor* b^{φ} between the corresponding Bénabou signatures). This functor, in its turn, will allow us to endow the many-sorted set $\mathrm{BT}_T(\Lambda)_{\varphi^{\sharp}\times\varphi^{\sharp}}$ with a Bénabou algebra structure for S, from which the definition of the composition of polyderivors will follow.

Proposition 4.11. Let φ be a mapping from S to $T^{\star}$. Then the $((S^{\star})^2)^{\star} \times (S^{\star})^2$-sorted mapping b^{φ} from Σ^{B_S} to $\mathrm{HT}_{T^{\star}\times T^{\star}}(\Sigma^{\mathrm{B}_T})_{(\varphi^{\sharp}\times\varphi^{\sharp})^{\star}\times(\varphi^{\sharp}\times\varphi^{\sharp})}$ defined as follows

1. For every $w \in S^{\star}$ and $\alpha \in |w|$, $b^{\varphi}(\pi^{w}_{\alpha})$ is the $\boldsymbol{\Sigma}^{\mathrm{B}_T}$-term

   $$\langle \pi^{\varphi^{\sharp}(w)}_{\sum_{\beta\in\alpha} p_{\beta}}, \dots, \pi^{\varphi^{\sharp}(w)}_{\sum_{\beta\in\alpha+1} p_{\beta}-1} \rangle_{\varphi^{\sharp}(w),\varphi(w_{\alpha})}$$

   of type $\lambda \longrightarrow (\varphi^{\sharp}(w), (\varphi(w_{\alpha})))$,
2. For every u, $w \in S^{\star}$, $b^{\varphi}(\langle\,\rangle_{u,w})$ is the $\boldsymbol{\Sigma}^{\mathrm{B}_T}$-term

   $$\begin{aligned}&\langle \pi^{\varphi(w_0)}_{0} \circ v^{(\varphi^{\sharp}(u),\varphi(w_0))}_{0}, \dots, \pi^{\varphi(w_0)}_{|\varphi(w_0)|-1} \circ v^{(\varphi^{\sharp}(u),\varphi(w_0))}_{0}, \dots,\\ &\pi^{\varphi(w_i)}_{0} \circ v^{(\varphi^{\sharp}(u),\varphi(w_i))}_{i}, \dots, \pi^{\varphi(w_i)}_{|\varphi(w_i)|-1} \circ v^{(\varphi^{\sharp}(u),\varphi(w_i))}_{i}, \dots,\\ &\pi^{\varphi(w_{|w|-1})}_{0} \circ v^{(\varphi^{\sharp}(u),\varphi(w_{|w|-1}))}_{|w|-1}, \dots, \pi^{\varphi(w_{|w|-1})}_{|\varphi(w_{|w|-1})|-1} \circ v^{(\varphi^{\sharp}(u),\varphi(w_{|w|-1}))}_{|w|-1} \rangle\end{aligned}$$

   of type $((\varphi^{\sharp}(u), \varphi(w_0)), \dots, (\varphi^{\sharp}(u), \varphi(w_{|w|-1}))) \longrightarrow (\varphi^{\sharp}(u), \varphi^{\sharp}(w))$,
3. For every u, x, $w \in S^{\star}$, $b^{\varphi}(\circ_{u,x,w})$ is the $\boldsymbol{\Sigma}^{\mathrm{B}_T}$-term

   $$\circ_{\varphi^{\sharp}(u),\varphi^{\sharp}(x),\varphi^{\sharp}(w)}(v^{(\varphi^{\sharp}(u),\varphi^{\sharp}(x))}_{0}, v^{(\varphi^{\sharp}(x),\varphi^{\sharp}(w))}_{1})$$

   of type $((\varphi^{\sharp}(u), \varphi^{\sharp}(x)), (\varphi^{\sharp}(x), \varphi^{\sharp}(w))) \longrightarrow (\varphi^{\sharp}(u), \varphi^{\sharp}(w))$,

is such that $(\varphi^{\sharp}\times\varphi^{\sharp}, b^{\varphi})\colon (S^{\star}\times S^{\star}, \Sigma^{\mathrm{B}_S}, \mathcal{E}^{\mathrm{B}_S}) \longrightarrow (T^{\star}\times T^{\star}, \Sigma^{\mathrm{B}_T}, \mathcal{E}^{\mathrm{B}_T})$ is a specification morphism. Therefore $\varphi\colon S \longrightarrow T^{\star}$ induces a functor $(\varphi^{\sharp} \times \varphi^{\sharp}, b^{\varphi})^{*}$ from $\mathbf{Alg}(\mathrm{B}_T)$ to $\mathbf{Alg}(\mathrm{B}_S)$ which sends $\mathbf{BT}_T(\Lambda)$, the free Bénabou algebra on the T-sorted signature Λ, to a Bénabou algebra for S, with $\mathrm{BT}_T(\Lambda)_{\varphi^{\sharp}\times\varphi^{\sharp}}$ as underlying $S^{\star} \times S^{\star}$-sorted set.

For a polyderivor $\mathbf{d}\colon \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$, we can extend the $S^{\star}\times S^{\star}$-sorted mapping d from $\coprod_{1\times\between_S} \Sigma$ to $\mathrm{BT}_T(\Lambda)_{\varphi^{\sharp}\times\varphi^{\sharp}}$ to a homomorphism of Bénabou algebras $d^{\sharp}$ from $\mathrm{BT}_S(\Sigma)$ to $\mathrm{BT}_T(\Lambda)_{\varphi^{\sharp}\times\varphi^{\sharp}}$, whose underlying $S^{\star} \times S^{\star}$-mapping determines a translation of terms for $\boldsymbol{\Sigma}$ into terms for $\boldsymbol{\Lambda}$.

We define next the composition of polyderivors and the identities.

Definition 4.12. Let $\mathbf{d}\colon \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ and $\mathbf{e}\colon \boldsymbol{\Lambda} \longrightarrow \boldsymbol{\Omega}$ be polyderivors. Then the *composition of* $\mathbf{d}$ *and* $\mathbf{e}$, denoted by $\mathbf{e} \circ \mathbf{d}$, is the morphism $(\psi^{\sharp} \circ \varphi, e^{\sharp}_{\varphi^{\sharp}\times\varphi^{\sharp}} \circ d)$, where the first component $\psi^{\sharp} \circ \varphi$ is a mapping from

S to $U^\star$ and $e^\sharp_{\varphi^\sharp\times\varphi^\sharp}\circ d$ is obtained from

$$\begin{tikzcd}
\coprod_{1\times\between_T}\Lambda \arrow[r, "\eta^{\mathrm{B}_T}_{\coprod_{1\times\between_T}\Lambda}"] \arrow[dr, "e"'] & \mathrm{BT}_T(\Lambda) \arrow[d, "e^\sharp"] & \text{as} & \mathrm{BT}_T(\Lambda)_{\varphi^\sharp\times\varphi^\sharp} \arrow[d, "e^\sharp_{\varphi^\sharp\times\varphi^\sharp}"] & \coprod_{1\times\between_S}\Sigma \arrow[l, "d"'] \\
 & \mathrm{BT}_U(\Omega)_{\psi^\sharp\times\psi^\sharp} & & \mathrm{BT}_U(\Omega)_{\psi^\sharp\times\psi^\sharp\,\varphi^\sharp\times\varphi^\sharp} &
\end{tikzcd}$$

On the other hand, for every signature $\boldsymbol{\Sigma}$, the *identity at* $\boldsymbol{\Sigma}$ is the polyderivor $(\between_S, \eta^{\mathrm{B}_S}_\Sigma)$.

From this definition we get the corresponding category of signatures and polyderivors.

Proposition 4.13. The signatures together with the polyderivors determine a category, that we denote by $\mathbf{Sig}_{\mathfrak{pd}}$.

Remark 11. From the fact that $\mathbf{Sig}_{\mathfrak{pd}}$ is a category it follows at once that, for every signature $\boldsymbol{\Sigma}$ in $\mathbf{Sig}_{\mathfrak{pd}}$, the set of all endopolyderivors of $\boldsymbol{\Sigma}$, $\mathrm{End}_{\mathfrak{pd}}(\boldsymbol{\Sigma})$, is the underlying set of a monoid, denoted by $\mathbf{End}_{\mathfrak{pd}}(\boldsymbol{\Sigma})$. Since the monoid $\mathbf{End}_{\mathfrak{d}}(\boldsymbol{\Sigma})$, of hypersubstitutions of $\boldsymbol{\Sigma}$, i.e, the monoid of endoderivors of $\boldsymbol{\Sigma}$, is embedded (in general, strictly) in the monoid $\mathbf{End}_{\mathfrak{pd}}(\boldsymbol{\Sigma})$, we conclude that $\mathbf{End}_{\mathfrak{pd}}(\boldsymbol{\Sigma})$ can serve as a basis to develop a doubly generalized (because of the use of many-sorted signatures and endopolyderivors, instead of single-sorted signatures and endoderivors,) theory of hyperidentities. But we leave this task for another occasion.

Having shown above that the concept of derivor, because of its reducibility to that of morphism of a Kleisli category for a monad in $\mathbf{Sig}$, is mathematically natural, one could also expect to prove the mathematical naturalness of the notion of polyderivor by proving that the category $\mathbf{Sig}_{\mathfrak{pd}}$ can be obtained as an isomorphic copy of the Kleisli category for a suitable monad in $\mathbf{Sig}$. This is actually true; however, the procedure which should be followed to determine such a monad is more complicated than the one, relatively simple, we have sketched for the derivors. This is due to the fact that, for a signature $\boldsymbol{\Sigma} = (S, \Sigma)$, the pair $(S^\star\times S^\star, \mathrm{BT}_S(\Sigma))$ is not a signature, because $\mathrm{BT}_S(\Sigma)$ is an $S^\star\times S^\star$-sorted set, but not an $((S^\star)^2)^\star\times(S^\star)^2$-sorted set. We notice that the proof of the existence of the monad in $\mathbf{Sig}$ whose Kleisli category is isomorphic to $\mathbf{Sig}_{\mathfrak{pd}}$ is founded, on the one hand, on the functor $\Delta_{\curlywedge_S\times 1}\colon \mathbf{Set}^{S^\star\times S^\star} \longrightarrow \mathbf{Set}^{S^{\star\star}\times S^\star}$ which sends $S^\star\times S^\star$-sorted sets to $S^\star$-signatures, therefore, for an S-sorted signature Σ, we will have that $\Delta_{\curlywedge_S\times 1}(\mathrm{BT}_S(\Sigma))$ is a $S^\star$-signature, and, on the other hand, of the fact that, for every set of sorts S, the adjunction $\mathbf{T}_{\mathrm{B}_S}\dashv \mathrm{G}_{\mathrm{B}_S}$, determines a monad $\mathbb{T}_{\mathrm{B}_S} = (\mathrm{T}_{\mathrm{B}_S}, \eta^{\mathrm{B}_S}, \mu^{\mathrm{B}_S})$ on $\mathbf{Set}^{S^\star\times S^\star}$.

Proposition 4.14. There exists a monad $\mathbb{T}_{\mathfrak{pd}} = (\mathfrak{pd}, \eta^{\mathfrak{pd}}, \mu^{\mathfrak{pd}})$ in $\mathbf{Sig}$ such that the categories $\mathbf{Sig}_{\mathfrak{pd}}$ and $\mathbf{Kl}(\mathbb{T}_{\mathfrak{pd}})$ are isomorphic.

Proof. Let $\mathfrak{pd}$ be the endofunctor of $\mathbf{Sig}$ defined as follows

1. $\mathfrak{pd}$ sends a signature $\boldsymbol{\Sigma}$ to $(S^\star, \mathrm{T}_{\mathrm{B}_S}(\coprod_{1\times\between_S}\Sigma)_{\curlywedge_S\times 1}))$.
2. $\mathfrak{pd}$ sends a signature morphism $\mathbf{d}$ from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$ to

   $$(\varphi^\star, (d^\sharp)_{\curlywedge_S\times 1})\colon (S^\star, \mathrm{T}_{\mathrm{B}_S}(\coprod_{1\times\between_S}\Sigma)_{\curlywedge_S\times 1})) \longrightarrow (T^\star, \mathrm{T}_{\mathrm{B}_T}(\coprod_{1\times\between_T}\Lambda)_{\curlywedge_T\times 1})),$$

   where $\mathrm{T}_{\mathrm{B}_S}(\coprod_{1\times\between_S}\Sigma)_{\curlywedge_S\times 1}$ is the value at Σ of the functor

$$\mathbf{Set}^{S^\star\times S} \xrightarrow{\coprod_{1\times\between_S}} \mathbf{Set}^{S^\star\times S^\star} \xrightarrow{\mathrm{T}_{\mathrm{B}_S}} \mathbf{Set}^{S^\star\times S^\star} \xrightarrow{\Delta_{\curlywedge_S\times 1}} \mathbf{Set}^{S^{\star\star}\times S^\star}.$$

After having defined the endofunctor $\mathfrak{pd}$ of $\mathbf{Sig}$, we proceed to define the unit $\eta^{\mathfrak{pd}}$ and multiplication $\mu^{\mathfrak{pd}}$ of the monad $\mathbb{T}_{\mathfrak{pd}}$.

Let $\boldsymbol{\Sigma}$ be a signature. Then we have that $\eta^{\mathfrak{pd}}_{\boldsymbol{\Sigma}}$, the component of the unit $\eta^{\mathfrak{pd}}$ of the purported monad $\mathbb{T}_{\mathfrak{pd}}$ in $\boldsymbol{\Sigma}$, is the signature morphism $(\between_S, \eta^{\mathrm{B}_S}_\Sigma)$, i.e., the value at Σ of the unit of the monad $\mathbb{T}_{\mathrm{B}_S} = (\mathrm{T}_{\mathrm{B}_S}, \eta^{\mathrm{B}_S}, \mu^{\mathrm{B}_S})$ in $\mathbf{Set}^{S^\star\times S^\star}$, obtained from the adjunction $\mathbf{T}_{\mathrm{B}_S}\dashv \mathrm{G}_{\mathrm{B}_S}$. On the other hand, we want $\mu^{\mathfrak{pd}}_{\boldsymbol{\Sigma}}$, the component of the multiplication $\mu^{\mathfrak{pd}}$ of the purported monad $\mathbb{T}_{\mathfrak{pd}}$ in $\mathbf{\Sigma}$, to be a morphism as in the following diagram

$$\begin{tikzcd}
(S^{\star\star}, \mathrm{T}_{\mathrm{B}_{S^{\star}}}(\coprod_{1\times\between_{S^{\star}}}(\mathrm{T}_{\mathrm{B}_{S}}(\coprod_{1\times\between_{S}}\Sigma)_{\curlywedge_{S}\times 1}))_{\curlywedge_{S^{\star}}\times 1}) \arrow[d, "\mu^{\mathfrak{pd}}_{\mathbf{\Sigma}}"'] \\
(S^{\star}, \mathrm{T}_{\mathrm{B}_{S}}(\coprod_{1\times\between_{S}}\Sigma)_{\curlywedge_{S}\times 1})
\end{tikzcd}$$

The first coordinate of $\mu^{\mathfrak{pd}}_{\mathbf{\Sigma}}$ is $\curlywedge_{S}$, the multiplication of the monad $\mathbb{T}_{\star}$. To get the second coordinate of $\mu^{\mathfrak{pd}}_{\mathbf{\Sigma}}$ we have to define a natural transformation α as in the following diagram

$$\begin{tikzcd}
 & \mathbf{Set}^{S^{\star\star}\times S^{\star\star}} \arrow[r, "\mathrm{T}_{\mathrm{B}_{S^{\star}}}"] & \mathbf{Set}^{S^{\star\star}\times S^{\star\star}} \arrow[r, "\Delta_{\curlywedge_{S^{\star}}\times 1}"] & \mathbf{Set}^{S^{\star\star\star}\times S^{\star\star}} \\
 & \mathbf{Set}^{S^{\star\star}\times S^{\star}} \arrow[u, "\coprod_{1\times\between_{S^{\star}}}"] \arrow[r, phantom, "\Searrow\alpha"] & {} \arrow[r, phantom, "="] & {} \\
\mathbf{Set}^{S^{\star}\times S^{\star}} \arrow[r, "\mathrm{T}_{\mathrm{B}_{S}}"] \arrow[rr, bend right=40, "\mathrm{T}_{\mathrm{B}_{S}}" description] & \mathbf{Set}^{S^{\star}\times S^{\star}} \arrow[r, "\mathrm{T}_{\mathrm{B}_{S}}"] \arrow[u, "\Delta_{\curlywedge_{S}\times 1}"] \arrow[d, phantom, "\Downarrow\mu^{\mathrm{B}_{S}}"] & \mathbf{Set}^{S^{\star}\times S^{\star}} \arrow[r, "\Delta_{\curlywedge_{S}\times 1}"] \arrow[uu, "\Delta_{\curlywedge_{S}\times\curlywedge_{S}}" description] & \mathbf{Set}^{S^{\star\star}\times S^{\star}} \arrow[uu, "\Delta_{\curlywedge_{S^{\star}}\times\curlywedge_{S}}" description] \\
\mathbf{Set}^{S^{\star}\times S} \arrow[u, "\coprod_{1\times\between_{S}}"] & {} & &
\end{tikzcd}$$

Let Θ be an $S^{\star}\times S^{\star}$-sorted set. Then $\mathrm{T}_{\mathrm{B}_{S}}(\Theta)_{\curlywedge_{S}\times\curlywedge_{S}}$ has a natural structure of $\mathbf{\Sigma}^{\mathrm{B}_{S^{\star}}}$-algebra, obtained from the $(S^{\star\star}\times S^{\star\star})^{\star}\times(S^{\star\star}\times S^{\star\star})$-sorted mapping $b^{\curlywedge_{S}}\colon \Sigma^{\mathrm{B}_{S^{\star}}} \longrightarrow \mathrm{Ter}_{S^{\star}\times S^{\star}}(\Sigma^{\mathrm{B}_{S}})_{(\curlywedge_{S}\times\curlywedge_{S})^{\star}\times(\curlywedge_{S}\times\curlywedge_{S})}$ by applying Proposition 4.11 to the mapping $\curlywedge_{S}\colon S^{\star\star}\longrightarrow S^{\star}$.

On the other hand, for every $S^{\star}\times S^{\star}$-sorted set Θ, we have an $S^{\star\star}\times S^{\star\star}$-sorted mapping f_{Θ} from $\coprod_{1\times\between_{S^{\star}}}(\Delta_{\curlywedge_{S}\times 1}(\Theta))$ to $\Delta_{\curlywedge_{S}\times\curlywedge_{S}}(\mathrm{T}_{\mathrm{B}_{S}}(\Theta))$ which, for every $(\overline{u},\overline{w})\in S^{\star\star}\times S^{\star\star}$, assigns to an element P, the image of P under the inclusion $\eta^{\mathrm{B}_{S}}_{\Theta}$ of Θ into $\mathrm{T}_{\mathrm{B}_{S}}(\Theta)$. The definition is sound because, in this case, $\overline{w}$ has the form (w), P is in $\Theta_{\curlywedge_{S}u,w}$ and $\eta^{\mathrm{B}_{S}}_{\Theta}(P)$ belongs to $\Delta_{\curlywedge_{S}\times\curlywedge_{S}}(\mathrm{T}_{\mathrm{B}_{S}}(\Theta))$. Then the extension $f^{\sharp}_{\Theta}$ of f_{Θ} to $\mathrm{T}_{\mathrm{B}_{S^{\star}}}(\coprod_{1\times\between_{S^{\star}}}(\Delta_{\curlywedge_{S}\times 1}(\Theta)))$ is the component at Θ of the natural transformation α.

Therefore, the second coordinate of $\mu^{\mathfrak{pd}}_{\mathbf{\Sigma}}$ is the value at Σ of the natural transformation

$$(\Delta_{\curlywedge_{S^{\star}}\times\curlywedge_{S}} * \Delta_{\curlywedge_{S}\times 1} * \mu^{\mathrm{B}_{S}} * \textstyle\coprod_{1\times\between_{S}}) \circ (\Delta_{\curlywedge_{S^{\star}}\times 1} * \alpha * \mathrm{T}_{\mathrm{B}_{S}} * \textstyle\coprod_{1\times\between_{S}}).$$

Finally we prove that the categories $\mathbf{Sig}_{\mathfrak{pd}}$ and $\mathbf{Kl}(\mathbb{T}_{\mathfrak{pd}})$ are isomorphic.

A morphism $\mathbf{d}\colon \mathbf{\Sigma}\longrightarrow\mathbf{\Lambda}$ in $\mathbf{Kl}(\mathbb{T}_{\mathfrak{pd}})$ is a morphism $\mathbf{d}\colon \mathbf{\Sigma}\longrightarrow\mathfrak{pd}(\mathbf{\Lambda})$ in $\mathbf{Sig}$, hence $\varphi\colon S\longrightarrow T^{\star}$ and

$$\begin{aligned}
d\colon \Sigma \longrightarrow &\ \Delta_{\varphi^{\star}\times\varphi}(\mathrm{T}_{\mathrm{B}_{T}}(\textstyle\coprod_{1\times\between_{T}}\Lambda)_{\curlywedge_{T}\times 1}) \\
&= \Delta_{\varphi^{\sharp}\times\varphi}(\mathrm{T}_{\mathrm{B}_{T}}(\textstyle\coprod_{1\times\between_{T}}\Lambda)) \\
&\cong \Delta_{\varphi^{\sharp}\times\varphi}(\mathrm{BT}_{T}(\Lambda)),
\end{aligned}$$

that is exactly the definition of polyderivor in $\mathbf{Sig}_{\mathfrak{pd}}$. □

Corollary 4.15. The category $\mathbf{Sig}_{\mathfrak{pd}}$ has coproducts.

Proof. This follows from Proposition 3.5, pp. 122–123, in [ClS05]. □

Our next goal is to lift the contravariant functor $\mathrm{Alg}\colon \mathbf{Sig}\longrightarrow\mathbf{Cat}$ to a contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}\colon \mathbf{Sig}_{\mathfrak{pd}}\longrightarrow\mathbf{Cat}$, that will allow us, by applying, once more, the EG-construction, to get a new category of algebras $\mathbf{Alg}_{\mathfrak{pd}}$ into which is embedded the category $\mathbf{Alg}$. But to achieve the just stated objective we should define beforehand some auxiliary functors and natural transformations.

Proposition 4.16. Let S be a set of sorts. Then we have that

1. There exists an *expansion* functor $(\cdot)^{\natural_S}$ from $\mathbf{Set}^S$ to $\mathbf{Set}^{S^\star}$ which sends an S-sorted set $A = (A_s)_{s\in S}$ to the $S^\star$-sorted set $A^{\natural_S} = (A_w)_{w\in S^\star}$, and an S-sorted mapping f from A to B to the $S^\star$-sorted mapping $f^{\natural_S} = (f_w)_{w\in S^\star}$ from $(A_w)_{w\in S^\star}$ to $(B_w)_{w\in S^\star}$. If A is an S-sorted set and $f\colon A \longrightarrow B$ an S-sorted mapping, then we say that $A^{\natural_S}$ and $f^{\natural_S}$ are the *expansions of* A and f, respectively, *to the words on* S and, to simplify the notation, we will write $A^\natural$ and $f^\natural$ instead of $A^{\natural_S}$ and $f^{\natural_S}$, respectively.
2. From the contravariant functor MSet, from $\mathbf{Set}$ to $\mathbf{Cat}$, to the contravariant functor $\mathrm{MSet}\circ \mathrm{T}_\star^{\mathrm{op}}$ between the same categories, where $\mathrm{T}_\star^{\mathrm{op}}$ is the composite of $\mathbf{T}_\star^{\mathrm{op}}$ (the dual of the free monoid functor $\mathbf{T}_\star$ from $\mathbf{Set}$ to $\mathbf{Mon}$, the category of monoids), and $G_{\mathbf{Mon}}$ (the forgetful functor from $\mathbf{Mon}$ to $\mathbf{Set}$), there exists a natural transformation $(\cdot)^\natural$ which sends a set S to the expansion functor $(\cdot)^{\natural_S}$ from $\mathbf{Set}^S$ to $\mathbf{Set}^{S^\star}$.
3. There exists a natural isomorphism ι_S from the functor $(\cdot)^{\natural_{S^\star}}\circ(\cdot)^{\natural_S}$ to the functor $\Delta_{\curlywedge_S}\circ(\cdot)^{\natural_S}$, both from the category $\mathbf{Set}^S$ to the category $\mathbf{Set}^{S^{\star\star}}$.

Proof. We restrict ourselves to prove the second and third parts of the proposition.

(2) $(\cdot)^\natural$ is a natural transformation from MSet to $\mathrm{MSet}\circ\mathrm{T}_\star^{\mathrm{op}}$ since, for a mapping $\varphi\colon S\longrightarrow T$, the functors $(\cdot)^{\natural_S}\circ\Delta_\varphi$ and $\Delta_{\varphi^\star}\circ(\cdot)^{\natural_T}$ from $\mathbf{Set}^T$ to $\mathbf{Set}^{S^\star}$ are identical. Observe, in particular, that for a T-sorted set B, we have that $(B_\varphi)^{\natural_S} = (B^{\natural_T})_{\varphi^\star}$.

(3) It is enough to define, for every S-sorted set A, the component $(\iota_S)_A$ of ι_S at A, as the $S^{\star\star}$-isomorphism $(\iota_S)_A\colon A^{\natural\natural}\longrightarrow (A^\natural)_\curlywedge$ that has as $\overline{w}$-th coordinate, for $\overline{w} = (w_\alpha)_{\alpha\in|\overline{w}|}\in S^{\star\star}$, the canonical isomorphism $\langle \mathrm{pr}_{\alpha_j}\circ\mathrm{pr}_\alpha\rangle_{\alpha\in|\overline{w}|, j\in|w_\alpha|}$ from $A^{\natural\natural}_{\overline{w}} = \prod_{\alpha\in|\overline{w}|}\prod_{j\in|w_\alpha|}A_{w_{\alpha_j}}$ to $A^\natural_{\curlywedge\overline{w}} = \prod_{\alpha\in|\overline{w}|,\, j\in|w_\alpha|}A_{w_{\alpha_j}}$, where $\mathrm{pr}_\alpha\colon A_{\overline{w}}\longrightarrow A_{w_\alpha}$ and $\mathrm{pr}_{\alpha_j}\colon A_{w_\alpha}\longrightarrow A_{w_{\alpha_j}}$ are the canonical projections. To simplify the notation we will write ι^A instead of $(\iota_S)_A$. ☐

Corollary 4.17. Let $\varphi\colon S\longrightarrow T^\star$ and $\psi\colon T\longrightarrow U^\star$ be mappings. Then, for every T-sorted set B and U-sorted set C, we have that

1. $((B^{\natural_T})^{\natural_{T^\star}})_{\varphi^\star}$, denoted by $B_{\varphi^\star}$, and $(B^{\natural_T})_{\varphi^\sharp}$, denoted by $B_{\varphi^\sharp}$, are isomorphic $S^\star$-sorted sets.
2. $(((C^{\natural_U})_\psi)^{\natural_T})_\varphi$, denoted by C_{ψ_φ}, and $(C^{\natural_U})_{\psi^\sharp\circ\varphi}$, denoted by $C_{\psi^\sharp\circ\varphi}$, are isomorphic S-sorted sets.
3. There exists an isomorphism $\kappa^B_\varphi\colon \mathrm{BO}_T(B)_{\varphi^\sharp\times\varphi^\sharp}\longrightarrow\mathrm{BO}_S(B_\varphi)$, where, to simplify, we have written B_φ instead of $(B^{\natural_T})_\varphi$.

We state in the following proposition that the polyderivors between signatures determine functors, in the opposite direction, from the category of algebras associated to the target signature to the category of algebras associated to the source signature. These functors will be the components of the morphism mapping of the contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$.

Proposition 4.18. Let $\mathbf{d}\colon\boldsymbol{\Sigma}\longrightarrow\boldsymbol{\Lambda}$ be a morphism in $\mathbf{Sig}_{\mathfrak{pd}}$. Then there exists a functor $\mathrm{Alg}_{\mathfrak{pd}}(\mathbf{d}) = \mathbf{d}^*_{\mathfrak{pd}}$ from $\mathbf{Alg}(\boldsymbol{\Lambda})$ to $\mathbf{Alg}(\boldsymbol{\Sigma})$. Its object mapping assigns to each $\boldsymbol{\Lambda}$-algebra $\mathbf{B} = (B, G)$ the $\boldsymbol{\Sigma}$-algebra $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B}) = (B_\varphi, G^{\mathbf{d}})$, where $G^{\mathbf{d}}$ is $\kappa^B_\varphi\circ G^\sharp_{\varphi^\sharp\times\varphi^\sharp}\circ d$, obtained from

$$\begin{tikzcd}
\coprod_{1\times\between_T}\Lambda \arrow[r, "\eta^{\mathrm{B}_T}_{\coprod_{1\times\between_T}\Lambda}"] \arrow[dr, "G"'] & \mathrm{BT}_T(\Lambda) \arrow[d, "G^\sharp"] \\
 & \mathrm{BO}_T(B)
\end{tikzcd}
\quad \text{as} \quad
\begin{tikzcd}
\mathrm{BT}_T(\Lambda)_{\varphi^\sharp\times\varphi^\sharp} \arrow[d, "G^\sharp_{\varphi^\sharp\times\varphi^\sharp}"'] & \coprod_{1\times\between_S}\Sigma \arrow[l, "d"'] \\
\mathrm{BO}_T(B)_{\varphi^\sharp\times\varphi^\sharp} \arrow[r, "\kappa^B_\varphi"'] & \mathrm{BO}_S(B_\varphi)
\end{tikzcd}$$

its morphism mapping assigns to each $\boldsymbol{\Lambda}$-homomorphism f from $\mathbf{B}$ to $\mathbf{B}'$ the $\boldsymbol{\Sigma}$-homomorphism $\mathbf{d}^*_{\mathfrak{pd}}(f) = f_\varphi$ from $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$ to $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B}')$.

Given a polyderivor $\mathbf{d}\colon\boldsymbol{\Sigma}\longrightarrow\boldsymbol{\Lambda}$, a $\boldsymbol{\Lambda}$-algebra $\mathbf{B} = (B, G)$ and an operation $\sigma\in\Sigma_{w,s}$, if we agree that w is the word $(s_i)_{i\in m}$, that, for every $i\in m$, $\varphi(s_i)$ is the word $(t_{i,j})_{j\in n_i}$, and that $\varphi(s)$ is the word $(t_k)_{k\in p}$, then we have that $\varphi^\sharp(w)$ is the word $(t_{0,0},\dots,t_{0,n_0-1},\dots,t_{m-1,0},\dots,t_{m-1,n_{m-1}-1})$ and that $d(\sigma)\colon \varphi^\sharp(w)\longrightarrow\varphi(s)$ is a family of terms $P=(P_0,\dots,P_{p-1})$ such that, for every $k\in p$, $P_k\colon \varphi^\sharp(w)\longrightarrow t_k$. Therefore $G^\sharp_{\varphi^\sharp\times\varphi^\sharp}(P)$, the realization of $d(\sigma)$ in $\mathbf{B}$, is precisely the term operation $P^{\mathbf{B}}=\langle P_0^{\mathbf{B}},\dots,P_{p-1}^{\mathbf{B}}\rangle$ of type

$$B_{t_{0,0}}\times\cdots\times B_{t_{0,n_0-1}}\times\cdots\times B_{t_{m-1,0}}\times\cdots\times B_{t_{m-1,n_{m-1}-1}}\longrightarrow B_{t_0}\times\cdots\times B_{t_{p-1}}$$

that by composition with the isomorphism from B_{φ_w} to $B_{\varphi^\sharp(w)}$ provides the operation $G^{\mathbf{d}}_\sigma$

$$\begin{tikzcd}
(B_{t_{0,0}}\times\cdots\times B_{t_{0,n_0-1}})\times\cdots\times(B_{t_{m-1,0}}\times\cdots\times B_{t_{m-1,n_{m-1}-1}}) \arrow[d, "\iota^B_{(\varphi(s_0),\dots,\varphi(s_{m-1}))}"] \\
B_{t_{0,0}}\times\cdots\times B_{t_{0,n_0-1}}\times\cdots\times B_{t_{m-1,0}}\times\cdots\times B_{t_{m-1,n_{m-1}-1}} \arrow[d, "P^{\mathbf{B}}"] \\
B_{t_0}\times\cdots\times B_{t_{p-1}}
\end{tikzcd}$$

It is now when we can properly state that the contravariant functor Alg from **Sig** to **Cat**, defined in the second section, can be lifted to a contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to **Cat**.

Proposition 4.19. There exists a contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category **Cat** given by the following data: its object mapping sends each signature $\mathbf{\Sigma}$ to $\mathrm{Alg}_{\mathfrak{pd}}(\mathbf{\Sigma})=\mathbf{Alg}(\mathbf{\Sigma})$; its arrow mapping sends each polyderivor $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ to $\mathbf{d}^*_{\mathfrak{pd}}\colon \mathbf{Alg}(\mathbf{\Lambda})\longrightarrow\mathbf{Alg}(\mathbf{\Sigma})$; for every $\mathbf{d}\colon\mathbf{\Sigma}\longrightarrow\mathbf{\Lambda}$ and $\mathbf{e}\colon\mathbf{\Lambda}\longrightarrow\mathbf{\Omega}$, the natural isomorphism $\gamma^{\mathbf{d},\mathbf{e}}$ from $\mathbf{e}^*_{\mathfrak{pd}}\circ\mathbf{d}^*_{\mathfrak{pd}}$ to $(\mathbf{e}\circ\mathbf{d})^*_{\mathfrak{pd}}$ is that which is defined, for every $\mathbf{\Omega}$-algebra $\mathbf{C}$, as the isomorphism $\iota^C_{\psi^\star\circ\varphi}$; for every $\mathbf{\Sigma}$, the natural isomorphism $\nu^{\mathbf{\Sigma}}$ from $\mathrm{Id}_{\mathbf{Alg}(\mathbf{\Sigma})}$ to $(\between_S,\eta^{\mathrm{B}_S}_{\mathbf{\Sigma}})^*_{\mathfrak{pd}}$ is that which is defined, for every $\mathbf{\Sigma}$-algebra $\mathbf{A}$, as the canonical isomorphism $\delta^A_S\colon A\longrightarrow(A_{(s)})_{s\in S}$.

By applying the EG-construction to the contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ we get the category $\mathbf{Alg}_{\mathfrak{pd}}$ as stated in the following definition.

Definition 4.20. The category $\mathbf{Alg}_{\mathfrak{pd}}$ is given by $\mathbf{Alg}_{\mathfrak{pd}}=\int^{\mathbf{Sig}_{\mathfrak{pd}}}\mathrm{Alg}_{\mathfrak{pd}}$. Therefore the category $\mathbf{Alg}_{\mathfrak{pd}}$ has as objects the pairs $(\mathbf{\Sigma},\mathbf{A})$, with $\mathbf{\Sigma}$ a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, and as morphisms from $(\mathbf{\Sigma},\mathbf{A})$ to $(\mathbf{\Lambda},\mathbf{B})$, the pairs $(\mathbf{d},h)$, with $\mathbf{d}$ a polyderivor from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and h a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$.

Example 5. Let $\mathbf{\Sigma}$ be a signature, $p\in\mathbb{N}$, and $\mathbf{d}=(\varphi,d)$ the endopolyderivor at $\mathbf{\Sigma}$, where $\varphi\colon S\longrightarrow S^\star$ is the mapping which sends $s\in S$ to the word $\curlywedge_{\mu\in p}(s)$ and, for $(w,s)\in S^\star\times S$, $d_{w,s}$ is the mapping from $\Sigma_{w,s}$ to $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi^\sharp(w))^p_s$ which sends $\sigma\in\Sigma_{w,s}$ to $(\sigma(v_0^{w_0},v_p^{w_1},\dots,v^{w_{|w|-1}}_{(|w|-1)p}),\dots,\sigma(v^{w_0}_{p-1},v^{w_1}_{2p-1},\dots,v^{w_{|w|-1}}_{|w|p-1}))$, in $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi^\sharp(w))^p_s$. Then, for the polyderivor $\mathbf{d}$ and two $\mathbf{\Sigma}$-algebras $\mathbf{A}$ and $\mathbf{B}$, we have that $(\mathbf{d},\langle h^\mu\rangle_{\mu\in p})$, where, for every $\mu\in p$, $h^\mu=(h^\mu_s)_{s\in S}$ is a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{B}$, is a morphism from $(\mathbf{\Sigma},\mathbf{A})$ to $(\mathbf{\Sigma},\mathbf{B})$, because $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})=\mathbf{B}^p$.

Additional examples related to computer sciences can be found in [GTW76].

The contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ is not only useful to construct the category $\mathbf{Alg}_{\mathfrak{pd}}$. Actually, as we prove in that which follows, it together with a pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ and a pseudo-extranatural transformation (Tr,θ) determine an institution $\mathfrak{Tm}_{\mathfrak{pd}}=(\mathbf{Sig}_{\mathfrak{pd}},\mathrm{Alg}_{\mathfrak{pd}},\mathrm{Ter}_{\mathfrak{pd}},(\mathrm{Tr},\theta))$ on **Set**, the so-called *many-sorted term institution of Fujiwara.*

We define next some auxiliary functors and natural transformations that we will use afterwards to prove, on the one hand, that there exists a pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from the category $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category **Cat**, which generalizes the pseudo-functor Ter from the category **Sig** to the 2-category **Cat**, and, on the other hand, that the category $\mathbf{Alg}_{\mathfrak{pd}}$ has coproducts.

Proposition 4.21. Let S be a set of sorts. Then we have that

1. There exists a *compression* functor $(\cdot)^{\dagger_S}$ from $\mathbf{Set}^{S^\star}$ to $\mathbf{Set}^S$, left adjoint to the expansion functor $(\cdot)^{\natural_S}$. Its object mapping sends an $S^\star$-sorted set C to the S-sorted set $C^{\dagger_S}=(\bigcup_{\substack{w\in S^\star\ \&\\ w^{-1}[s]\neq\varnothing}}(C_w\times\{w\}\times w^{-1}[s]))_{s\in S}$;

its morphism mapping assigns to each $S^\star$-mapping $f\colon C \longrightarrow C'$ the S-sorted mapping $f^{\dagger_S}\colon C^{\dagger_S} \longrightarrow C'^{\dagger_S}$ which, for each $s \in S$, sends each (c, w, i) in $C^{\dagger_S}_s$ to $f^{\dagger_S}_s(c, w, i) = (f_w(c), w, i)$ in $C'^{\dagger_S}_s$

2. From the contravariant functor $\mathrm{MSet} \circ \mathrm{T}^{\mathrm{op}}_\star$, from **Set** to **Cat**, to the contravariant functor MSet between the same categories, there exists a natural transformation $(\cdot)^\dagger$ which sends a set S to the compression functor $(\cdot)^{\dagger_S}$ from $\mathbf{Set}^{S^\star}$ to $\mathbf{Set}^S$.
3. There exists a natural isomorphism ζ_S from the functor $(\cdot)^{\dagger_S} \circ (\cdot)^{\dagger_{S^\star}}$ to the functor $(\cdot)^{\dagger_S} \circ \coprod_{⅄_S}$.

Proof. We restrict ourselves to prove the first and third part of the proposition.

(1) For every $S^\star$-sorted set C and S-sorted set A, there exists a natural isomorphism $\theta^{\dagger\natural}\colon \mathrm{Hom}(C^{\dagger_S}, A) \cong \mathrm{Hom}(C, A^{\natural_S})$ which assigns to an S-sorted mapping $f\colon C^{\dagger_S} \longrightarrow A$ the $S^\star$- sorted mapping $\theta^{\dagger\natural}(f)$, defined, for every $w \in S^\star$ and $c \in C_w$, as $\theta^{\dagger\natural}(f)_w(c) = (f_{w_i}(c, w, i))_{i\in|w|}$.

Reciprocally, if $g\colon C \longrightarrow A^{\natural_S}$ is an $S^\star$-sorted mapping, then $(\theta^{\dagger\natural})^{-1}(g)$ is the S-sorted mapping defined, for every $s \in S$ and $(c, w, i) \in C^{\dagger_S}_s$, as $(\theta^{\dagger\natural})^{-1}(g)_s(c, w, i) = g_w(c)_i$.

(3) By Proposition 4.16, the functors $(\cdot)^{\natural_{S^\star}} \circ (\cdot)^{\natural_S}$ and $\Delta_{⅄_S} \circ (\cdot)^{\natural_S}$ are isomorphic. Furthermore, $(\cdot)^{\dagger_S} \circ (\cdot)^{\dagger_{S^\star}}$ is left adjoint to $(\cdot)^{\natural_{S^\star}} \circ (\cdot)^{\natural_S}$ and $(\cdot)^{\dagger_S} \circ \coprod_{⅄_S}$ is left adjoint to $\Delta_{⅄_S} \circ (\cdot)^{\natural_S}$, thus the functors $(\cdot)^{\dagger_S} \circ (\cdot)^{\dagger_{S^\star}}$ and $(\cdot)^{\dagger_S} \circ \coprod_{⅄_S}$ are isomorphic. We denote such a natural isomorphism by ζ_S, and, to simplify the notation, we will write ζ^C instead of $(\zeta_S)_C$ for the component of ζ_S at C. □

If $\varphi\colon S \longrightarrow T^\star$ is a mapping, then from the adjunctions $\coprod_\varphi \dashv \Delta_\varphi$ and $(\cdot)^{\dagger_T} \dashv (\cdot)^{\natural_T}$, we get the adjunction $\coprod^\dagger_\varphi \dashv \Delta^\natural_\varphi$ where, to simplify the notation, we have written $\coprod^\dagger_\varphi$ instead of $(\cdot)^{\dagger_T} \circ \coprod_\varphi$ and $\Delta^\natural_\varphi$ instead of $\Delta_\varphi \circ (\cdot)^{\natural_T}$. Furthermore, we agree that $\theta^{\dagger\natural}_\varphi$, $\eta^{\dagger\natural}_\varphi$, and $\varepsilon^{\dagger\natural}_\varphi$ denote, respectively, the natural isomorphism, the unit, and the counit of this composite adjunction.

What we want to establish now is that the category $\mathbf{Alg}_{\mathfrak{pd}}$ has coproducts and for this we begin by proving that, for every polyderivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the functor $\mathbf{d}^*_{\mathfrak{pd}}$ from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$ has a left adjoint $\mathbf{d}^{\mathfrak{pd}}_*$.

Proposition 4.22. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a polyderivor. Then there exists a functor $\mathbf{d}^{\mathfrak{pd}}_*$ from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Alg}(\mathbf{\Lambda})$ that is left adjoint to the functor $\mathbf{d}^*_{\mathfrak{pd}}$ from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$.

Proof. We restrict ourselves to define the action of $\mathbf{d}^{\mathfrak{pd}}_*$ on the objects (because the remaining details being like those of Proposition 2.16, although more cumbersome, they can be left to the reader). Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra. Then $\mathbf{d}^{\mathfrak{pd}}_*(\mathbf{A})$ is the $\mathbf{\Lambda}$-algebra defined as $\mathbf{T}_{\mathbf{\Lambda}}(\coprod^\dagger_\varphi A)/\overline{R}^{\mathbf{A}}$, where $\overline{R}^{\mathbf{A}}$ is the congruence on $\mathbf{T}_{\mathbf{\Lambda}}(\coprod^\dagger_\varphi A)$ generated by the T-sorted relation $R^{\mathbf{A}}$, defined, for every $t \in T$, as

$$R^{\mathbf{A}}_t = \left\{ \left((F^{\mathbf{A}}_\sigma(a_i \mid i \in |w|), s, \varphi(s), j), d(\sigma)_j(\mathbf{a})\right) \;\middle|\; \begin{array}{c} j \in \varphi(s)^{-1}[t],\ w \in S^\star, \\ s \in S,\ \sigma \in \Sigma_{w,s},\ a \in A_w \end{array} \right\},$$

$\mathbf{a}$ being the matrix

$$\mathbf{a} = \begin{pmatrix} {\scriptstyle (a_0, w_0, \varphi(w_0), 0)} & \cdots & {\scriptstyle (a_0, w_0, \varphi(w_0), |\varphi(w_0)|-1)} \\ \vdots & \ddots & \vdots \\ {\scriptstyle (a_{|w|-1}, w_{|w|-1}, \varphi(w_{|w|-1}), 0)} & \cdots & {\scriptstyle (a_{|w|-1}, w_{|w|-1}, \varphi(w_{|w|-1}), |\varphi(w_{|w|-1})|-1)} \end{pmatrix},$$

and $d(\sigma)_j(\mathbf{a})$ the result of replacing the variables in the term $d(\sigma)_j$ with the entries in the matrix $\mathbf{a}$ (recall that, for $\sigma \in \Sigma_{w,s}$, we have agreed that $d(\sigma) = d_{w,s}(\sigma)$, where $d_{w,s}(\sigma) \in \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^\sharp(w))_{\varphi(s)}$, hence, for every $j \in |\varphi(s)|$, $d(\sigma)_j \in \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^\sharp(w))_{\varphi(s)_j}$). □

Proposition 4.23. The category $\mathbf{Alg}_{\mathfrak{pd}}$ has coproducts.

Proof. The category $\mathbf{Sig}_{\mathfrak{pd}}$ has coproducts, for every signature $\mathbf{\Sigma}$, the category $\mathbf{Alg}(\mathbf{\Sigma})$ has coproducts, and the functor $\mathrm{Alg}_{\mathfrak{pd}}$ is locally reversible. Therefore, from Theorem 2, pp. 250–251, in [TBG91], the category $\mathbf{Sig}_{\mathfrak{pd}}$ has coproducts. □

Our next goal is to state that every polyderivor induces a functor between the associated categories of terms as was the case for the signature morphisms.

Proposition 4.24. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a polyderivor. Then there exists a functor $\mathbf{d}^{\mathfrak{pd}}_{\diamond}$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})$. Its object mapping assigns to each S-sorted set X the T-sorted set $\mathbf{d}^{\mathfrak{pd}}_{\diamond}(X) = \coprod^{\dagger}_{\varphi} X$; its morphism mapping assigns to each morphism P from X to Y in $\mathbf{Ter}(\mathbf{\Sigma})$ the morphism $\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P) = (\theta^{\dagger\natural}_{\varphi})^{-1}(\eta^{\mathbf{d}}_{X} \circ P)$ from $\coprod^{\dagger}_{\varphi} X$ to $\coprod^{\dagger}_{\varphi} Y$, where $\theta^{\dagger\natural}_{\varphi}$ is the natural isomorphism of the adjunction $\coprod^{\dagger}_{\varphi} \dashv \Delta^{\natural}_{\varphi}$, $\eta^{\mathbf{d}}_{X}$ the $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\Delta^{\natural}_{\varphi}(\mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X))$ that extends the S-sorted mapping $\Delta^{\natural}_{\varphi}(\eta_{\coprod^{\dagger}_{\varphi} X}) \circ (\eta^{\dagger\natural}_{\varphi})_X$ from X to $\Delta^{\natural}_{\varphi}(\mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X))$, and $\eta^{\dagger\natural}_{\varphi}$ the unit of the adjunction $\coprod^{\dagger}_{\varphi} \dashv \Delta^{\natural}_{\varphi}$.

Proof. The proof is founded on the fact that, for every term $P\colon X \longrightarrow Y$, the term $\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)\colon \coprod^{\dagger}_{\varphi} X \longrightarrow \coprod^{\dagger}_{\varphi} Y$ is the composition of the morphisms in the following diagram

$$\begin{tikzcd}
\coprod^{\dagger}_{\varphi} Y \arrow[r, "\coprod^{\dagger}_{\varphi} P"] \arrow[rrd, "\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)"'] & \coprod^{\dagger}_{\varphi} \mathrm{T}_{\mathbf{\Sigma}}(X) \arrow[r, "\coprod^{\dagger}_{\varphi} \eta^{\mathbf{d}}_{X}"] & \coprod^{\dagger}_{\varphi} \Delta^{\natural}_{\varphi}(\mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)) \arrow[d, "(\varepsilon^{\dagger\natural}_{\varphi})_{\mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)}"] \\
 & & \mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)
\end{tikzcd}$$

where $(\varepsilon^{\dagger\natural}_{\varphi})_{\mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)}$ is the value at $\mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)$ of the counit of the adjunction $\coprod^{\dagger}_{\varphi} \dashv \Delta^{\natural}_{\varphi}$. □

Remark 12. In Proposition 4.24, $(\eta^{\dagger\natural}_{\varphi})_X$, the value at X of the unit of the adjunction $\coprod^{\dagger}_{\varphi} \dashv \Delta^{\natural}_{\varphi}$, in its s-th coordinate, assigns to $x \in X_s$, the family $((x, s, \varphi(s), 0), \ldots, (x, s, \varphi(s), |\varphi(s)| - 1))$. We can say, informally, that to a variable $x \in X_s$ corresponds in $\coprod^{\dagger}_{\varphi} X$ a family of variables of the form $(x, s, \varphi(s), i)$, with sorts $\varphi(s)_i$, and that, for a morphism $P\colon X \longrightarrow Y$ in $\mathbf{Ter}(\mathbf{\Sigma})$ and $(y, s, \varphi(s), i) \in (\coprod^{\dagger}_{\varphi} Y)_t$, $\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)_{\varphi(s)_i}(y, s, \varphi(s), i)$ is the term for $\mathbf{\Lambda}$ obtained by replacing, recursively, in $P_s(y)$ the formal operations $\sigma\colon w \longrightarrow s$ by the families of formal operations $d(\sigma)\colon \varphi^{\sharp}(w) \longrightarrow \varphi(s)$ and the variables $x \in X_s$ by families of variables $(x, s, \varphi(s), j)_{j \in |\varphi(s)|}$.

Before we state that the above construction can be lifted to a pseudo-functor from the category $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$, we point out that the relation of satisfaction is also invariant under polyderivor change, i.e., that for every polyderivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, if (P, Q) is a $\mathbf{\Sigma}$-equation of type (X, Y) and $\mathbf{A}$ a $\mathbf{\Lambda}$-algebra, then

$$\mathbf{d}^{*}_{\mathfrak{pd}}(\mathbf{A}) \models^{\mathbf{\Sigma}}_{X,Y} (P, Q) \text{ iff } \mathbf{A} \models^{\mathbf{\Lambda}}_{\coprod^{\dagger}_{\varphi} X, \coprod^{\dagger}_{\varphi} Y} (\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P), \mathbf{d}^{\mathfrak{pd}}_{\diamond}(Q)).$$

This follows from the invariant character under signature change through the polyderivors of the realization of terms as term operations in arbitrary, but fixed, algebras, as stated in the following proposition.

Proposition 4.25. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a polyderivor. Then, for every $\mathbf{\Lambda}$-algebra $\mathbf{A}$ and term $P\colon X \longrightarrow Y$ for $\mathbf{\Sigma}$ of type (X, Y), the following diagram commutes

$$\begin{tikzcd}
(A_{\varphi})_X \arrow[r, "P^{\mathbf{d}^{*}_{\mathfrak{pd}}(\mathbf{A})}"] & (A_{\varphi})_Y \\
A_{\coprod^{\dagger}_{\varphi} X} \arrow[u, "(\theta^{\dagger\natural}_{\varphi})_{X,A}"] \arrow[r, "\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)^{\mathbf{A}}"'] & A_{\coprod^{\dagger}_{\varphi} Y} \arrow[u, "(\theta^{\dagger\natural}_{\varphi})_{Y,A}"']
\end{tikzcd}$$

where, to simplify the notation, we have written A_{φ} instead of the more accurate $\Delta^{\natural}_{\varphi} A$.

Proof. The proof is analogous to that of Proposition 2.25. □

It is now when we can properly state that the pseudo-functor Ter from $\mathbf{Sig}$ to the 2-category $\mathbf{Cat}$, defined in the second section, can be lifted to a pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$.

Proposition 4.26. There exists a pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category **Cat** given by the following data

1. The object mapping of $\mathrm{Ter}_{\mathfrak{pd}}$ is that which sends $\mathbf{\Sigma}$ in $\mathbf{Sig}_{\mathfrak{pd}}$ to $\mathrm{Ter}_{\mathfrak{pd}}(\mathbf{\Sigma}) = \mathbf{Ter}(\mathbf{\Sigma})$.
2. The morphism mapping of $\mathrm{Ter}_{\mathfrak{pd}}$ is that which sends $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ in $\mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{d}^{\mathfrak{pd}}_{\diamond}\colon \mathbf{Ter}(\mathbf{\Sigma}) \longrightarrow \mathbf{Ter}(\mathbf{\Lambda})$.
3. For $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ and $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$, the natural isomorphism $\gamma^{\mathbf{d},\mathbf{e}}$ from the composite $\mathbf{e}^{\mathfrak{pd}}_{\diamond} \circ \mathbf{d}^{\mathfrak{pd}}_{\diamond}$ to $(\mathbf{e} \circ \mathbf{d})^{\mathfrak{pd}}_{\diamond}$ is that which is defined, for every S-sorted set X, as the isomorphism $\gamma^{\mathbf{d},\mathbf{e}}_X\colon \coprod^{\dagger}_{\psi} \coprod^{\dagger}_{\varphi} X \longrightarrow \coprod^{\dagger}_{\psi^{\sharp}\circ\varphi} X$ in $\mathbf{Ter}(\mathbf{\Omega})$ that corresponds to the U-sorted mapping
$$\coprod\nolimits^{\dagger}_{\psi^{\sharp}\circ\varphi} X \xrightarrow{\ \rho_X\ } \coprod\nolimits^{\dagger}_{\psi} \coprod\nolimits^{\dagger}_{\varphi} X \xrightarrow{\ \eta_{\coprod^{\dagger}_{\psi} \coprod^{\dagger}_{\varphi} X}\ } \mathrm{T}_{\mathbf{\Omega}}(\coprod\nolimits^{\dagger}_{\psi} \coprod\nolimits^{\dagger}_{\varphi} X),$$
where ρ is the isomorphism obtained from the following diagram
$$\begin{tikzcd}
\mathbf{Set}^{S} \arrow[d, "\coprod^{\dagger}_{\varphi}"'] \arrow[dr, "\coprod_{\varphi}"] \arrow[dddr, bend left=70, "\coprod_{\psi^{\sharp}\circ\varphi}"] & & & \\
\mathbf{Set}^{T} \arrow[d, "\coprod^{\dagger}_{\psi}"'] \arrow[dr, "\coprod_{\psi}"] & \mathbf{Set}^{T^{\star}} \arrow[l, "(\cdot)^{\dagger_T}"'] \arrow[dr, "\coprod_{\psi^{\star}}"] \arrow[d, phantom, "="] \arrow[dd, bend left=60, "\coprod_{\psi^{\sharp}}"] & & \\
\mathbf{Set}^{U} & \mathbf{Set}^{U^{\star}} \arrow[l, "(\cdot)^{\dagger_U}"'] \arrow[d, phantom, "\Uparrow(\zeta^{U^{\star}})^{-1}"] & \mathbf{Set}^{U^{\star\star}} \arrow[l, "(\cdot)^{\dagger_{U^{\star}}}"'] \arrow[dl, "\coprod_{\curlywedge_U}"] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "(\gamma^{\varphi,\psi^{\sharp}})^{-1}\Swarrow"] & {} \\
& \mathbf{Set}^{U^{\star}} \arrow[ul, "(\cdot)^{\dagger_U}"] & &
\end{tikzcd}$$
and γ the isomorphism associated to the pseudo-functor $\mathrm{MSet}^{\amalg}$.
4. For $\mathbf{\Sigma}$, the natural isomorphism $\nu^{\mathbf{\Sigma}}$ from $\mathrm{Id}_{\mathbf{Ter}(\mathbf{\Sigma})}$ to $(\between_S, \eta^{\mathrm{B}_S}_{\Sigma})^{\mathfrak{pd}}_{\diamond}$ is that which is defined, for an S-sorted set X, as the isomorphism $\nu^{\mathbf{\Sigma}}_X$ from X to $\coprod^{\dagger}_{\between_S} X$ that corresponds to the S-sorted mapping $\eta_X \circ \tau^S_X$ from $\coprod^{\dagger}_{\between_S} X$ to $\mathrm{T}_{\mathbf{\Omega}}(X)$, where τ^S is the natural isomorphism from $(\cdot)^{\dagger_S} \circ \coprod_{\between_S}$ to $\mathrm{Id}_{\mathbf{Set}^S}$ defined, for an S-sorted set X, as the S-sorted mapping whose s-th coordinate, for $s \in S$, sends an $((a,s),(s),0) \in (\coprod^{\dagger}_{\between_S} X)_s$ to $(\tau^S_X)_s((a,s),(s),0) = a$.

Following this we state a lemma from which we will get a pseudo-extranatural transformation that formalizes the invariant character of the realization of terms in algebras relative to the polyderivors between signatures.

Lemma 4.27. Let $\mathbf{\Sigma}$ be a signature and $\widetilde{\mathbf{id}}_{\mathfrak{pd}(\mathbf{\Sigma})} = (\mathrm{id}_{S^{\star}}, \widetilde{\mathrm{id}}_{\mathfrak{pd}(\mathbf{\Sigma})})$ the polyderivor from $\mathfrak{pd}(\mathbf{\Sigma})$ to $\mathbf{\Sigma}$, where $\mathrm{id}_{S^{\star}}\colon S^{\star} \longrightarrow S^{\star}$ is the identity at $S^{\star}$ while $\widetilde{\mathrm{id}}_{\mathfrak{pd}(\mathbf{\Sigma})}$ is the canonical isomorphism from $\mathrm{T}_{\mathrm{B}_S}(\coprod_{1\times\between_S}\Sigma)_{\curlywedge_S\times 1}$ to $\mathrm{BT}_S(\coprod_{1\times\between_S}\Sigma)_{\curlywedge_S\times 1}$. Then the family $(\theta^{\dagger\natural}_{X,A})_{(\mathbf{A},X)\in\mathbf{Alg}(\mathbf{\Sigma})\times\mathbf{Ter}(\mathfrak{pd}(\mathbf{\Sigma}))}$ is a natural isomorphism as shown in the following diagram
$$\begin{tikzcd}
\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathfrak{pd}(\mathbf{\Sigma})) \arrow[r, "\alpha_{\mathbf{\Sigma}} \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \beta_{\mathbf{\Sigma}}"'] \arrow[dr, phantom, "\overset{\theta^{\dagger\natural}}{\Nearrow}"] & \mathbf{Alg}(\mathfrak{pd}(\mathbf{\Sigma})) \times \mathbf{Ter}(\mathfrak{pd}(\mathbf{\Sigma})) \arrow[d, "\mathrm{Tr}^{\mathfrak{pd}(\mathbf{\Sigma})}"] \\
\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma}) \arrow[r, "\mathrm{Tr}^{\mathbf{\Sigma}}"'] & \mathbf{Set}
\end{tikzcd}$$

where, to abbreviate, $\alpha_{\mathbf{\Sigma}} = (\widetilde{\mathbf{id}}_{\mathfrak{pd}(\mathbf{\Sigma})})^{*}_{\mathfrak{pd}}$ and $\beta_{\mathbf{\Sigma}} = (\widetilde{\mathbf{id}}_{\mathfrak{pd}(\mathbf{\Sigma})})^{\mathfrak{pd}}_{\diamond}$.

Proof. By Proposition 2.35 we have that Tr is a pseudo-extranatural transformation, hence, for the morphism $\widetilde{\mathbf{id}}_{\mathfrak{pd}(\mathbf{\Sigma})} \colon \mathfrak{pd}(\mathbf{\Sigma}) \longrightarrow \mathbf{\Sigma}$, the above diagram commutes. In particular, for a morphism $(f, P) \colon (\mathbf{A}, X) \longrightarrow (\mathbf{B}, Y)$ in the product category $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathfrak{pd}(\mathbf{\Sigma}))$, we have the configuration

$$\begin{tikzcd}
& (\mathbf{A},X) \arrow[rr, "{(f,P)}"] \arrow[dl, mapsto] & & (\mathbf{B},Y) \arrow[dr, mapsto] & & \\
(\mathbf{A},X^{\dagger}) \arrow[r, "{(f,\beta_{\mathbf{\Sigma}}(P))}"] & (\mathbf{B},Y^{\dagger}) \arrow[dd, mapsto] & & & (\alpha_{\mathbf{\Sigma}}(\mathbf{A}),X) \arrow[r, "{(f^{\natural},P)}"] & (\alpha_{\mathbf{\Sigma}}(\mathbf{B}),Y) \arrow[dl, mapsto] \\
& & & & A^{\natural}_{X} \arrow[ddl, "f^{\natural}_{X}"'] \arrow[ddr, "P^{\alpha_{\mathbf{\Sigma}}(\mathbf{A})}"] & \\
& A_{X^{\dagger}} \arrow[urrr, "\theta^{\dagger\natural}_{X,A}"] \arrow[ddl, "f_{X^{\dagger}}"'] \arrow[ddr, "\beta_{\mathbf{\Sigma}}(P)^{\mathbf{A}}"] & & & & \\
& & & B^{\natural}_{X} \arrow[ddr, "P^{\alpha_{\mathbf{\Sigma}}(\mathbf{B})}"] & & A^{\natural}_{Y} \arrow[ddl, "f^{\natural}_{Y}"] \\
B_{X^{\dagger}} \arrow[urrr, "\theta^{\dagger\natural}_{X,B}"] \arrow[ddr, "\beta_{\mathbf{\Sigma}}(P)^{\mathbf{B}}"'] & & A_{Y^{\dagger}} \arrow[urrr, "\theta^{\dagger\natural}_{Y,A}"] \arrow[ddl, "f_{Y^{\dagger}}"'] & & & \\
& & & & B^{\natural}_{Y} & \\
& B_{Y^{\dagger}} \arrow[urrr, "\theta^{\dagger\natural}_{Y,B}"] & & & &
\end{tikzcd}$$

and $(\theta^{\dagger\natural}_{X,A})_{(\mathbf{A},X)\in\mathbf{Alg}(\mathbf{\Sigma})\times\mathbf{Ter}(\mathfrak{pd}(\mathbf{\Sigma}))}$ is a natural isomorphism. □

Remark 13. The functors $\alpha_{\mathbf{\Sigma}} = (\widetilde{\mathbf{id}}_{\mathfrak{pd}(\mathbf{\Sigma})})^{*}_{\mathfrak{pd}}$ from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Alg}(\mathfrak{pd}(\mathbf{\Sigma}))$ are the components of a natural transformation α from Alg into Alg $\circ\, \mathfrak{pd}^{\mathrm{op}}$, both from $\mathbf{Sig}^{\mathrm{op}}$ to $\mathbf{Cat}$. In its turn, the functors $\beta_{\mathbf{\Sigma}} = (\widetilde{\mathbf{id}}_{\mathfrak{pd}(\mathbf{\Sigma})})^{\mathfrak{pd}}_{\diamond}$ from $\mathbf{Ter}(\mathfrak{pd}(\mathbf{\Sigma}))$ to $\mathbf{Ter}(\mathbf{\Sigma})$ are the components of a natural transformation β from Ter $\circ\, \mathfrak{pd}$ into Ter, both from $\mathbf{Sig}$ to $\mathbf{Cat}$. Besides, if for a polyderivor $\mathbf{d} \colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ we denote by $\widetilde{\mathbf{d}} \colon \mathbf{\Sigma} \longrightarrow \mathfrak{pd}(\mathbf{\Lambda})$ the signature morphism associated to $\mathbf{d}$, by the isomorphism between $\mathbf{Sig}_{\mathfrak{pd}}$ and $\mathbf{Kl}(\mathbb{T}_{\mathfrak{pd}})$ stated in Proposition 4.14, then we have that

1. $\mathbf{d}^{*}_{\mathfrak{pd}} = \widetilde{\mathbf{d}}^{*} \circ \alpha_{\mathbf{\Lambda}}$, and
2. $\mathbf{d}^{\mathfrak{pd}}_{\diamond} = \beta_{\mathbf{\Lambda}} \circ \widetilde{\mathbf{d}}_{\diamond}$.

Therefore the morphism mappings of the pseudo-functors $\mathrm{Alg}_{\mathfrak{pd}}$ and $\mathrm{Ter}_{\mathfrak{pd}}$ are definable through the natural transformations α and β, respectively.

In the following proposition we construct a pseudo-functor from the product category $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$, and prove that there exists a pseudo-extranatural transformation from it to the functor from the product category $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$ that is constantly $\mathbf{Set}$.

Proposition 4.28. There exists a pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}_{\mathfrak{pd}}(\cdot)$ from the category $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$, obtained from the pseudo-functors $\mathrm{Alg}_{\mathfrak{pd}}$ and $\mathrm{Ter}_{\mathfrak{pd}}$, which sends a pair of signatures $(\mathbf{\Sigma}, \mathbf{\Lambda})$ to the category $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Lambda})$, and a pair of signature morphisms $(\mathbf{d}, \mathbf{e})$ from $(\mathbf{\Sigma}, \mathbf{\Lambda})$ to $(\mathbf{\Sigma}', \mathbf{\Lambda}')$ in $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to the functor $\mathbf{d}^{*}_{\mathfrak{pd}} \times \mathbf{d}^{\mathfrak{pd}}_{\diamond}$ from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma}') \times \mathbf{Ter}(\mathbf{\Lambda}')$. Furthermore, the family of functors $\mathrm{Tr} = (\mathrm{Tr}^{\mathbf{\Sigma}})_{\mathbf{\Sigma}\in\mathbf{Sig}_{\mathfrak{pd}}}$, together with the family $\theta = (\theta^{\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Sig}_{\mathfrak{pd}})}$, with $\theta^{\mathbf{d}}_{\mathbf{A},X} = \theta^{\dagger\natural}_{X,A}$, is a pseudo-extranatural transformation from the pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}_{\mathfrak{pd}}(\cdot)$ to the functor $\mathrm{K}_{\mathbf{Set}}$ from $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$ which picks $\mathbf{Set}$.

Proof. We restrict ourselves to prove that, for every polyderivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the following diagram iso-commutes.

$$\begin{tikzcd}
\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma}) \arrow[r, "\mathbf{d}^{*}_{\mathfrak{p}\mathfrak{d}} \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \mathbf{d}^{\mathfrak{p}\mathfrak{d}}_{\diamond}"'] & \mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma}) \arrow[d, "\mathrm{Tr}^{\mathbf{\Sigma}}"] \\
\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Lambda}) \arrow[r, "\mathrm{Tr}^{\mathbf{\Lambda}}"'] & \mathbf{Set}
\end{tikzcd}$$

But in the following diagram, where, we recall, $\widetilde{\mathbf{d}}\colon \mathbf{\Sigma} \longrightarrow \mathfrak{p}\mathfrak{d}(\mathbf{\Lambda})$ is the signature morphism associated to the polyderivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, by Proposition 4.14,

$$\begin{tikzcd}
\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma}) \arrow[rrr, "\mathbf{d}^{*}_{\mathfrak{p}\mathfrak{d}} \times \mathrm{Id}"] \arrow[ddd, "\mathrm{Id} \times \mathbf{d}^{\mathfrak{p}\mathfrak{d}}_{\diamond}"'] \arrow[ddr, "\mathrm{Id} \times \widetilde{\mathbf{d}}_{\diamond}"'] \arrow[drr, "\alpha_{\mathbf{\Lambda}} \times \mathrm{Id}"] & & & \mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma}) \arrow[ddd, "\mathrm{Tr}^{\mathbf{\Sigma}}"] \\
 & & \mathbf{Alg}(\mathfrak{p}\mathfrak{d}(\mathbf{\Lambda})) \times \mathbf{Ter}(\mathbf{\Sigma}) \arrow[ur, "\widetilde{\mathbf{d}}^{*} \times \mathrm{Id}"'] \arrow[d, "\mathrm{Id} \times \widetilde{\mathbf{d}}_{\diamond}"'] \arrow[dr, phantom, "(2)"] & \\
 & \mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathfrak{p}\mathfrak{d}(\mathbf{\Lambda})) \arrow[r, "\alpha_{\mathbf{\Lambda}} \times \mathrm{Id}"] \arrow[dl, "\mathrm{Id} \times \beta_{\mathbf{\Lambda}}"] \arrow[drr, phantom, "(1)"] & \mathbf{Alg}(\mathfrak{p}\mathfrak{d}(\mathbf{\Lambda})) \times \mathbf{Ter}(\mathfrak{p}\mathfrak{d}(\mathbf{\Lambda})) \arrow[dr, "\mathrm{Tr}^{\mathfrak{p}\mathfrak{d}(\mathbf{\Lambda})}"'] & \\
\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Lambda}) \arrow[rrr, "\mathrm{Tr}^{\mathbf{\Lambda}}"'] & & & \mathbf{Set}
\end{tikzcd}$$

we have that the bottom trapezoid (1) iso-commutes by Lemma 4.27, the right-hand trapezoid (2) iso-commutes because Tr is a pseudo-extranatural transformation and the remaining subdiagrams commute by the definitions of the involved entities. □

Corollary 4.29. The quadruple $\mathfrak{Tm}_{\mathfrak{p}\mathfrak{d}} = (\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}, \mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}, \mathrm{Ter}_{\mathfrak{p}\mathfrak{d}}, (\mathrm{Tr}, \theta))$ is an institution on **Set**, the so-called *many-sorted term institution of Fujiwara*, or, to abbreviate, the *term institution of Fujiwara.*

Remark 14. Since every standard signature morphism can be identified to a polyderivor, $\mathfrak{Tm}$, the term institution is canonically embedded into $\mathfrak{Tm}_{\mathfrak{p}\mathfrak{d}}$, the term institution of Fujiwara.

We close this section by pointing out that the replicas of Definition 3.4 and Corollary 3.5 are also valid for the category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$.

5. Transformations of Fujiwara and the term 2-institution of Fujiwara.

What we ultimately try to do is to define (once we have at our disposal a convenient notion of morphism from a specification into another, but for polyderivors), for two morphisms $\mathbf{d}$, $\mathbf{e}$ from a specification $(\mathbf{\Sigma}, \mathcal{E})$ to another $(\mathbf{\Lambda}, \mathcal{H})$, a concept of transformation from $\mathbf{d}$ to $\mathbf{e}$, as well as vertical and horizontal compositions for these transformations, in such a way that specifications, morphisms, and transformations constitute a 2-category $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$. But to succeed in doing it we should begin, as we do in this section, by defining a 2-category structure on the category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$, of signatures and polyderivors, through the concept of transformation between polyderivors.

In this section we also prove that the transformations between polyderivors determine natural transformations between the functors associated to the polyderivors, that allow us to lift the pseudo-functors $\mathrm{Alg}_{\mathfrak{pd}}$ and $\mathrm{Ter}_{\mathfrak{pd}}$ to 2-functors, and hence to get, by applying an EG-construction to $\mathrm{Alg}_{\mathfrak{pd}}$, a 2-category $\mathbf{Alg}_{\mathfrak{pd}}$. Besides, we prove that the transformations between polyderivors are also compatible with the realization of the terms in the algebras and we characterize this through the concept of pseudo-extranatural transformation between pseudo-functors on 2-categories. From this we get that the relation between terms and algebras is an example of 2-institution.

In order to define and investigate the transformations between polyderivors it will be shown to be convenient to make use of some derived operations in the Bénabou algebras of terms for the different signatures, concretely of those in the following definition.

Definition 5.1. Let S be a set of sorts.

1. For every $\overline{w} \in S^{\star\star}$ and $\alpha \in |\overline{w}|$, let $\pi^{\overline{w}}_{\alpha}$ be the derived operation of type $\lambda \longrightarrow (\curlywedge\overline{w}, \overline{w}_{\alpha})$ defined as

   $$\langle \pi^{\curlywedge\overline{w}}_{\sum_{\beta\in\alpha} p_\beta}, \dots, \pi^{\curlywedge\overline{w}}_{\sum_{\beta\in\alpha+1} p_\beta - 1} \rangle_{\curlywedge\overline{w}, \overline{w}_\alpha},$$

   where $\overline{w}$ is of the form

   $$((\cdot, \dots, \cdot), \dots, \overbrace{(\underset{\sum_{\beta\in\alpha} p_\beta}{\cdot}, \dots, \underset{\sum_{\beta\in\alpha+1} p_\beta - 1}{\cdot})}^{\overline{w}_\alpha}, \dots, (\cdot, \dots, \cdot)),$$

   and, for every $\alpha \in |\overline{w}|$, $p_\alpha = |\overline{w}_\alpha|$.
2. For every $u \in S^{\star}$ and $\overline{w} \in S^{\star\star}$, let $\langle\,\rangle_{u,\overline{w}}$ be the derived operation of type

   $$((u, \overline{w}_0), \dots, (u, \overline{w}_{|\overline{w}|-1})) \longrightarrow (u, \curlywedge\overline{w})$$

   defined as

   $$\begin{aligned}\langle P_0, \dots, P_{|\overline{w}|-1} \rangle_{u,\overline{w}} = \langle & \pi^{\overline{w}_0}_0 \circ P_0, \dots, \pi^{\overline{w}_0}_{|\overline{w}_0|-1} \circ P_0, \dots, \\ & \pi^{\overline{w}_{|\overline{w}|-1}}_0 \circ P_{|\overline{w}|-1}, \dots, \pi^{\overline{w}_{|\overline{w}|-1}}_{|\overline{w}_{|\overline{w}|-1}|-1} \circ P_{|\overline{w}|-1} \rangle_{u,\curlywedge\overline{w}}.\end{aligned}$$

3. For every $n \in \mathbb{N}$, and $\overline{u}$, $\overline{w} \in S^{\star n}$, let $\bigcurlywedge_{\overline{u},\overline{w}}$ be the derived operation of type

   $$((\overline{u}_0, \overline{w}_0), \dots, (\overline{u}_{n-1}, \overline{w}_{n-1})) \longrightarrow (\bigcurlywedge\overline{u}, \bigcurlywedge\overline{w})$$

   defined as

   $$\bigcurlywedge_{\overline{u},\overline{w}}(P_0, \dots, P_{n-1}) = \langle P_0 \circ \pi^{\overline{u}}_0, \dots, P_{n-1} \circ \pi^{\overline{u}}_{n-1} \rangle_{\curlywedge\overline{u},\overline{w}}.$$

Henceforth, to simplify the notation, we will omit some subscripts in the expressions. Moreover, for the operations of the form $\bigcurlywedge_{\overline{u},\overline{w}}$ we adopt the infix notation, and we will write $P_0 \curlywedge \cdots \curlywedge P_{n-1}$ instead of $\bigcurlywedge_{\overline{u},\overline{w}}(P_0, \dots, P_{n-1})$, the type, in its turn, will be $\overline{u}_0 \curlywedge \cdots \curlywedge \overline{u}_{n-1} \longrightarrow \overline{w}_0 \curlywedge \cdots \curlywedge \overline{w}_{n-1}$.

For the algebras of terms $\mathbf{BT}_S(\Sigma)$, the operations $\bigcurlywedge_{\overline{u},\overline{w}}$ are, essentially, the result of gathering into a family the corresponding terms, relabelling adequately the variables.

Recalling that the Bénabou algebras are, up to isomorphism, the finitary many-sorted algebraic theories of Bénabou, we will represent, henceforth, the composition of terms diagrammatically, and the equality of two coterminal paths composed of terms by asserting the commutativity of the appropriate diagram.

Definition 5.2. Let $\mathbf{d}$ and $\mathbf{e}$ be polyderivors from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$. A *transformation from* $\mathbf{d}$ *to* $\mathbf{e}$ is a choice function ξ for the family $(\mathrm{BT}_T(\Lambda)_{\varphi(s),\psi(s)})_{s\in S} = (\mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi(s))_{\psi(s)})_{s\in S}$, such that, for every operation $\sigma\colon w \longrightarrow s$, the following diagram commutes

$$\begin{tikzcd}
1 \arrow[r, "\langle \xi_s, d(\sigma) \rangle"] \arrow[d, "\langle e(\sigma), \xi_w \rangle"'] & \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi(s))_{\psi(s)} \times \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^{\sharp}(w))_{\varphi(s)} \arrow[d, "\circ"] \\
\mathrm{T}_{\mathbf{\Lambda}}(\downarrow\psi^{\sharp}(w))_{\psi(s)} \times \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^{\sharp}(w))_{\psi^{\sharp}(w)} \arrow[r, "\circ"'] & \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^{\sharp}(w))_{\psi(s)}
\end{tikzcd}$$

or more briefly, such that $\xi_s \circ d(\sigma) = e(\sigma) \circ \xi_w$, where ξ_w is $\xi_{w_0} \curlywedge \cdots \curlywedge \xi_{w_{|w|-1}}$. From now on, we will use the symbol $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ to denote that ξ is a transformation from $\mathbf{d}$ to $\mathbf{e}$.

Therefore for a transformation $\xi = (\xi_s)_{s\in S}$ from $\mathbf{d}$ to $\mathbf{e}$ we have, in particular, that, for every $s \in S$, $\xi_s \in \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi(s))_{\psi(s)}$, i.e., that $\xi_s = ((\xi_s)_0, \ldots, (\xi_s)_{|\psi(s)|-1})$ is a tuple of length $|\psi(s)|$ such that, for every $i \in |\psi(s)|$, $(\xi_s)_i$ is a term for $\mathbf{\Lambda}$ of type $(\downarrow\varphi(s), \psi(s)_i)$.

Henceforth, we shall write the above commutativity condition for a transformation $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ between polyderivors in the form $e(\sigma) \circ \xi_w = \xi_s \circ d(\sigma)$.

Example 6. Let $\mathbf{\Sigma}$ be a signature, $p, q \in \mathbb{N}$, $\mathbf{d} = (\varphi, d)$ the endopolyderivor of $\mathbf{\Sigma}$, where $\varphi\colon S \longrightarrow S^\star$ is the mapping which sends $s \in S$ to the word $\curlywedge_{\mu\in p}(s)$ and, for $(w, s) \in S^\star \times S$, $d_{w,s}$ the mapping from $\Sigma_{w,s}$ to $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi^\sharp(w))^p_s$ which sends $\sigma \in \Sigma_{w,s}$ to $(\sigma(v_0^{w_0}, v_p^{w_1}, \ldots, v_{(|w|-1)p}^{w_{|w|-1}}), \ldots, \sigma(v_{p-1}^{w_0}, v_{2p-1}^{w_1}, \ldots, v_{|w|p-1}^{w_{|w|-1}}))$, and $\mathbf{e} = (\psi, e)$ the endopolyderivor of $\mathbf{\Sigma}$, where $\psi\colon S \longrightarrow S^\star$ is the mapping which sends $s \in S$ to the word $\curlywedge_{\nu\in q}(s)$ and, for $(w, s) \in S^\star \times S$, $e_{w,s}$ the mapping from $\Sigma_{w,s}$ to $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\psi^\sharp(w))^q_s$ which sends $\sigma \in \Sigma_{w,s}$ to $(\sigma(v_0^{w_0}, v_q^{w_1}, \ldots, v_{(|w|-1)q}^{w_{|w|-1}}), \ldots, \sigma(v_{q-1}^{w_0}, v_{2q-1}^{w_1}, \ldots, v_{|w|q-1}^{w_{|w|-1}}))$. Then, for an arbitrary, but fixed, mapping $f = (f(\nu))_{\nu\in q}$ from the natural number q to the natural number p, taking as ξ the element of $\prod_{s\in S} \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi(s))^q_s$ defined, for every $s \in S$, as $\xi_s = (v^s_{f(0)}, \ldots, v^s_{f(q-1)})$, where, to simplify notation, we have identified the variables in $\downarrow\varphi(s)$ with their images in $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi(s))$ under $\eta_{\downarrow\varphi(s)}$, we have that ξ is a transformation from $\mathbf{d}$ to $\mathbf{e}$. We point out that the working out of all the details of this example, even if a little troublesome, helps to grasp the functioning of the polyderivors and the transformations between them.

The commutativity condition in the above definition of transformation from a polyderivor into another can be extended to the terms, as proved in the following proposition.

Proposition 5.3. Let $\mathbf{d}$ and $\mathbf{e}$ be polyderivors from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ a transformation. Then, for every term $P\colon u \longrightarrow w$ in $\mathrm{BT}_S(\Sigma)$, we have the following equation $\xi_w \circ d^\sharp(P) = e^\sharp(P) \circ \xi_u$.

Proof. By algebraic induction in the Bénabou algebra $\mathbf{BT}_S(\Sigma)$. $\square$

What we want now is to endow the category $\mathbf{Sig}_{\mathfrak{pd}}$ of signatures and polyderivors with a 2-category structure. For this we provide in the following proposition the definitions of the horizontal and vertical composition of the transformations between polyderivors, prove the law of Godement, and define the identity transformations at the polyderivors.

Proposition 5.4. The signatures together with the polyderivors and the transformations between the polyderivors have a 2-category structure, denoted as $\mathbf{Sig}_{\mathfrak{pd}}$.

Proof. Definition of the vertical composition. Given the configuration

$$\begin{tikzcd}
 & \mathbf{d} \arrow[dr, bend left] \arrow[d, squiggly, "\xi"] & \\
\mathbf{\Sigma} \arrow[ur, bend left, no head] \arrow[r, no head] \arrow[dr, bend right, no head] & \mathbf{e} \arrow[r] \arrow[d, squiggly, "\chi"] & \mathbf{\Lambda} \\
 & \mathbf{h} \arrow[ur, bend right] &
\end{tikzcd}$$

the vertical composition of ξ and χ, defined as $\chi \circ \xi = (\chi_s \circ \xi_s)_{s\in S}$, is a transformation from $\mathbf{d}$ to $\mathbf{h}$.

Definition of the horizontal composition. Given the configuration

$$\begin{tikzcd}
 & \mathbf{d} \arrow[dr, bend left] \arrow[dd, squiggly, "\xi"] & & \mathbf{h} \arrow[dr, bend left] \arrow[dd, squiggly, "\chi"] & \\
\mathbf{\Sigma} \arrow[ur, bend left, no head] \arrow[dr, bend right, no head] & & \mathbf{\Lambda} \arrow[ur, bend left, no head] \arrow[dr, bend right, no head] & & \mathbf{\Omega} \\
 & \mathbf{e} \arrow[ur, bend right] & & \mathbf{i} \arrow[ur, bend right] &
\end{tikzcd}$$

the horizontal composition of ξ and χ, defined as $\chi * \xi = (\chi_{\psi(s)} \circ h^\sharp(\xi_s))_{s\in S}$, or, equivalently, as $(i^\sharp(\xi_s) \circ \chi_{\varphi(s)})_{s\in S}$, is a transformation from $\mathbf{h} \circ \mathbf{d}$ to $\mathbf{i} \circ \mathbf{e}$. We have to prove that $\chi * \xi$ is a transformation from $(\gamma^\sharp \circ \varphi, h^\sharp_{\varphi^\sharp\times\varphi^\sharp} \circ d)$ to $(\nu^\sharp \circ \psi, i^\sharp_{\psi^\sharp\times\psi^\sharp} \circ e)$, i.e., that, for every $\sigma\colon w \longrightarrow s$, we have that $(\chi * \xi)_s \circ h^\sharp(d(\sigma)) = i^\sharp(e(\sigma)) \circ (\chi * \xi)_w$. But this happens since ξ, χ are transformations and $h^\sharp$, $i^\sharp$ morphisms.

Law of Godement. Given the configuration

$$\begin{tikzcd}
 & \mathbf{d}_0 \arrow[dr] \arrow[d, squiggly, "\xi"] & & \mathbf{e}_0 \arrow[dr] \arrow[d, squiggly, "\xi'"] & \\
\mathbf{\Sigma} \arrow[ur, bend left, dash] \arrow[r, dash] \arrow[dr, bend right, dash] & \mathbf{d}_1 \arrow[r] \arrow[d, squiggly, "\chi"] & \mathbf{\Lambda} \arrow[ur, bend left, dash] \arrow[r, dash] \arrow[dr, bend right, dash] & \mathbf{e}_1 \arrow[r] \arrow[d, squiggly, "\chi'"] & \mathbf{\Omega} \\
 & \mathbf{d}_2 \arrow[ur, bend right] & & \mathbf{e}_2 \arrow[ur, bend right] &
\end{tikzcd}$$

we have, after the definitions of the vertical and horizontal compositions, that

$$(\chi' * \chi) \circ (\xi' * \xi) = (\chi' \circ \xi') * (\chi \circ \xi).$$

Identities. Finally, given polyderivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ and $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$ it is obvious that the S-family

$$(\langle \pi^{\varphi(s)}_0, \ldots, \pi^{\varphi(s)}_{|\varphi(s)|-1} \rangle_{\varphi(s),\varphi(s)})_{s \in S},$$

denoted by $\mathrm{id}_{\mathbf{d}}$, is the identity transformation at $\mathbf{d}$, and that $\mathrm{id}_{\mathbf{e}} * \mathrm{id}_{\mathbf{d}} = \mathrm{id}_{\mathbf{e} \circ \mathbf{d}}$. □

Our next goal is to prove that the transformations between polyderivors from a signature into another, determine natural transformations between the functors between the categories of algebras associated to the signatures. To accomplish this we begin by proving that every transformation ξ from a polyderivor $\mathbf{d}$ to another one $\mathbf{e}$, both from a signature $\mathbf{\Sigma}$ to a signature $\mathbf{\Lambda}$, determines, for a given $\mathbf{\Lambda}$-algebra $\mathbf{B}$, a $\mathbf{\Sigma}$-homomorphism $\xi^{\mathbf{B}}$ from $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$ to $\mathbf{e}^*_{\mathfrak{pd}}(\mathbf{B})$.

Proposition 5.5. Let $\mathbf{d}$ and $\mathbf{e}$ be polyderivors from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ a transformation in $\mathbf{Sig}_{\mathfrak{pd}}$, and, for a $\mathbf{\Lambda}$-algebra $\mathbf{B} = (B, G)$, let $\xi^{\mathbf{B}}$ be the S-sorted mapping $(\xi^{\mathbf{B}}_s)_{s \in S}$ from B_φ to B_ψ, where, for every $s \in S$, we have that $\xi^{\mathbf{B}}_s = G^\sharp_{\varphi(s),\psi(s)}(\xi_s)\colon B_{\varphi(s)} \longrightarrow B_{\psi(s)}$. Then $\xi^{\mathbf{B}}$ is a $\mathbf{\Sigma}$-homomorphism from $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$ to $\mathbf{e}^*_{\mathfrak{pd}}(\mathbf{B})$.

Proof. For every operation $\sigma\colon w \longrightarrow s$, in Σ, we have to prove that $G^{\mathbf{e}}_\sigma \circ \xi^{\mathbf{B}}_w = \xi^{\mathbf{B}}_s \circ G^{\mathbf{d}}_\sigma$, and for this it is enough to prove that every face, except at most the frontal one, in the following diagram commutes

$$\begin{tikzcd}
 & B_{\varphi^\sharp(w)} \arrow[rr, "(G^\sharp_{\varphi^\sharp \times \varphi^\sharp} \circ d)_{w,s}(\sigma)"] \arrow[dd, "(\xi_w)^{\mathbf{B}}"'] & & B_{\varphi^\sharp(s)} \arrow[dr, "(\iota^B_{\varphi^\star(s)})^{-1}"] \arrow[dd, "(\xi_s)^{\mathbf{B}}"] & \\
B_{\varphi_w} \arrow[ur, "\iota^B_{\varphi^\star(w)}"] \arrow[rrrr, "G^{\mathbf{d}}_\sigma"] \arrow[dd, "(\xi^{\mathbf{B}})_w"'] & & & & B_{\varphi_s} \arrow[dd, "(\xi^{\mathbf{B}})_s"] \\
 & B_{\psi^\sharp(w)} \arrow[rr, "(G^\sharp_{\psi^\sharp \times \psi^\sharp} \circ e)_{w,s}(\sigma)"'] & & B_{\psi^\sharp(s)} \arrow[dr, "(\iota^B_{\psi^\star(s)})^{-1}"] & \\
B_{\psi_w} \arrow[ur, "\iota^B_{\psi^\star(w)}"] \arrow[rrrr, "G^{\mathbf{e}}_\sigma"'] & & & & B_{\psi_s}
\end{tikzcd}$$

from which it follows, necessarily, that the frontal face also commutes. The details are left to the reader. □

After having proved, for two polyderivors $\mathbf{d}$ and $\mathbf{e}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, that every transformation ξ from $\mathbf{d}$ to $\mathbf{e}$, induces, for every $\mathbf{\Lambda}$-algebra $\mathbf{B}$, a $\mathbf{\Sigma}$-homomorphism $\xi^{\mathbf{B}}$ from $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$ to $\mathbf{e}^*_{\mathfrak{pd}}(\mathbf{B})$, we prove in the following proposition the naturalness of the involved procedure.

Proposition 5.6. Let $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ be a transformation with $\mathbf{d}$ and $\mathbf{e}$ polyderivors from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$. Then the family $(\xi^{\mathbf{B}})_{\mathbf{B} \in \mathbf{Alg}(\mathbf{\Lambda})}$, denoted by $\mathrm{Alg}_{\mathfrak{pd}}(\xi)$, is a natural transformation from the functor $\mathbf{d}^*_{\mathfrak{pd}}$ to the functor $\mathbf{e}^*_{\mathfrak{pd}}$, both from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$.

Proof. We have to prove that, for every $\mathbf{\Lambda}$-algebras $\mathbf{B} = (B, G)$, $\mathbf{C} = (C, H)$ and morphism $f\colon \mathbf{B} \longrightarrow \mathbf{C}$ in $\mathbf{Alg}(\mathbf{\Lambda})$, the $\mathbf{\Sigma}$-homomorphism $\xi^{\mathbf{C}} \circ f_\varphi$ and $f_\psi \circ \xi^{\mathbf{B}}$ from $(B_\varphi, G^{\mathbf{d}})$ to $(C_\psi, H^{\mathbf{e}})$ are identical. But this is immediate since, for every $s \in S$, $\xi^{\mathbf{B}}_s$ and $\xi^{\mathbf{C}}_s$ being the realizations of the term ξ_s in the respective algebras, the mappings $\xi^{\mathbf{C}}_s \circ f_{\varphi(s)}$ and $f_{\psi(s)} \circ \xi^{\mathbf{B}}_s$ from $B_{\varphi(s)}$ to $C_{\psi(s)}$ necessarily coincide. □

Once stated that the transformations between polyderivors from a signature into another, induce natural transformations among the functors between the categories of algebras associated to the signatures, we can properly lift the pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}\colon \mathbf{Sig}_{\mathfrak{pd}} \longrightarrow \mathbf{Cat}$ to the 2-cells in the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$.

Proposition 5.7. There exists a pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$, contravariant in the morphisms and covariant in the 2-cells, from the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$, together with the accompanying natural isomorphisms $\gamma^{\mathbf{d},\mathbf{e}}$ and $\nu^{\mathbf{\Sigma}}$, as defined in Proposition 4.19.

Proof. It follows from the fact that the natural isomorphisms of the pseudo-functor are compatible with the 2-category structure of $\mathbf{Sig}_{\mathfrak{pd}}$. □

On the basis of this last proposition we can lift the category $\mathbf{Alg}_{\mathfrak{pd}}$ to a 2-category as in the following definition.

Definition 5.8. Let $\mathbf{Alg}_{\mathfrak{pd}} = \iint^{\mathbf{Sig}_{\mathfrak{pd}}} \mathrm{Alg}_{\mathfrak{pd}}$ be the 2-category which has as objects (0-cells) the pairs $(\mathbf{\Sigma}, \mathbf{A})$, where $\mathbf{\Sigma}$ is a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra; as morphisms (1-cells) from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$ the pairs $(\mathbf{d}, f)$, where $\mathbf{d}$ is a polyderivor from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and f a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^{*}_{\mathfrak{pd}}(\mathbf{B})$; and as 2-cells from $(\mathbf{d}, f)$ to $(\mathbf{e}, g)$, where $(\mathbf{d}, f)$ and $(\mathbf{e}, g)$ are morphisms from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the 2-cells $\xi\colon \mathbf{\Sigma} \rightsquigarrow \mathbf{\Lambda}$ in $\mathbf{Sig}_{\mathfrak{pd}}$ such that $\xi^{\mathbf{B}} \circ f = g$.

As was the case above for algebras and transformations, our goal now is to prove that the transformations between polyderivors from a signature into another, also determine natural transformations between the functors between the categories of terms associated to the signatures. To accomplish this we begin by proving that every transformation ξ from a polyderivor $\mathbf{d}$ to another one $\mathbf{e}$, both from a signature $\mathbf{\Sigma}$ to a signature $\mathbf{\Lambda}$, determines, for a given S-sorted set X, a morphism ξ_X, in the category $\mathbf{Ter}(\mathbf{\Lambda})$, from $\coprod^{\dagger}_{\varphi} X$ to $\coprod^{\dagger}_{\psi} X$.

Proposition 5.9. Let $\mathbf{d}$ and $\mathbf{e}$ be polyderivors from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ a transformation in $\mathbf{Sig}_{\mathfrak{pd}}$, and, for an S-sorted set X, let $\xi_X\colon \coprod^{\dagger}_{\psi} X \longrightarrow \mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)$ be the T-sorted mapping defined, for each $t \in T$ and each $(x, s, \psi(s), i) \in (\coprod^{\dagger}_{\psi} X)_t$, as follows

$$(\xi_X)_t(x, s, \psi(s), i) = (\xi_s)_i(v_j^{\varphi(s)_j}/(x, s, \varphi(s), j) \mid j \in |\varphi(s)|).$$

Then the mapping ξ_X is a morphism, in the category $\mathbf{Ter}(\mathbf{\Lambda})$, from $\coprod^{\dagger}_{\varphi} X$ to $\coprod^{\dagger}_{\psi} X$.

Proof. The definition of $\xi_X\colon \coprod^{\dagger}_{\psi} X \longrightarrow \mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)$ is sound since, for every j in $|\varphi(s)|$, we have that $(x, s, \varphi(s), j) \in (\coprod^{\dagger}_{\varphi} X)_{\varphi(s)_j}$ and $(\xi_s)_i \in \mathrm{T}_{\mathbf{\Lambda}}(\varphi(s))_{\psi(s)_i}$, hence $(\xi_X)_t(x, s, \varphi(s), i)$ is a term for $\mathbf{\Lambda}$ of type $\psi(s)_i = t$. □

After having proved, for two polyderivors $\mathbf{d}$ and $\mathbf{e}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, that every transformation ξ from $\mathbf{d}$ to $\mathbf{e}$, induces, for every S-sorted set X, a morphism ξ_X from $\coprod^{\dagger}_{\varphi} X$ to $\coprod^{\dagger}_{\psi} X$, we prove in the following proposition that they are the components of a natural transformation.

Proposition 5.10. Let $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ be a transformation in $\mathbf{Sig}_{\mathfrak{pd}}$, with $\mathbf{d}$, $\mathbf{e}$ polyderivors from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$. Then $\mathrm{Ter}_{\mathfrak{pd}}(\xi) = (\xi_X)_{X \in \mathbf{Ter}(\mathbf{\Sigma})}$ is a natural transformation from $\mathbf{d}^{\mathfrak{pd}}_{\diamond}$ to $\mathbf{e}^{\mathfrak{pd}}_{\diamond}$.

Proof. Because, for a morphism $P\colon X \longrightarrow Y$ in $\mathbf{Ter}(\mathbf{\Sigma})$, the T-sorted mappings $\xi_Y \circ \mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)$ and $\mathbf{e}^{\mathfrak{pd}}_{\diamond}(P) \circ \xi_X$ from $\coprod^{\dagger}_{\varphi} X$ to $\coprod^{\dagger}_{\psi} Y$ are identical. □

Let us observe that this last proposition is analogous to Proposition 5.3 but for derived operations with variables in arbitrary many-sorted sets.

Once stated that the transformations between polyderivors from a signature into another, induce natural transformations among the functors between the categories of terms associated to the signatures, we can properly lift the pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}\colon \mathbf{Sig}_{\mathfrak{pd}} \longrightarrow \mathbf{Cat}$ to the 2-cells of the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$.

Proposition 5.11. There exists a pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$, covariant in the morphisms and the 2-cells, together with the accompanying natural isomorphisms $\gamma^{\mathbf{d},\mathbf{e}}$ and $\nu^{\mathbf{\Sigma}}$, as defined in Proposition 4.26.

Proof. It follows from the fact that the natural isomorphisms of the pseudo-functor are compatible with the 2-category structure of $\mathbf{Sig}_{\mathfrak{pd}}$. $\square$

Since $\mathbf{Sig}_{\mathfrak{pd}}$ is a 2-category, our next goal will be to prove that $\mathfrak{Tm}_{\mathfrak{pd}}$ is not only an institution on **Set** but actually a 2-institution on **Set**. To attain the just stated goal we should begin by proving that the realization of the terms in the algebras is compatible with the additional structure derived from the 2-cells in $\mathbf{Sig}_{\mathfrak{pd}}$, i.e., the transformations between polyderivors.

Lemma 5.12. Let $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ be a transformation in $\mathbf{Sig}_{\mathfrak{pd}}$ from the polyderivor $\mathbf{d}$ to the polyderivor $\mathbf{e}$, both from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$. Then, for every $\mathbf{\Lambda}$-algebra $\mathbf{B}$, and S-sorted set X, the mappings $(\xi^{\mathbf{B}})_X \circ (\theta^{\dagger\natural}_{\varphi})_{X,B}$ and $(\theta^{\dagger\natural}_{\psi})_{X,B} \circ (\xi_X)^{\mathbf{B}}$ from $B_{\coprod^{\dagger}_{\varphi} X}$ to $(\Delta^{\natural}_{\psi} B)_X$ are identical.

Proof. For every $f \in B_{\coprod^{\dagger}_{\varphi} X}$, $(\xi_X)^{\mathbf{B}}(f) \in B_{\coprod^{\dagger}_{\psi} X}$ is the morphism $f^{\sharp} \circ \xi_X$, where $f^{\sharp}$ is the extension of f to $\mathbf{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)$, obtained as shown in the following diagram

$$\begin{tikzcd}
\coprod^{\dagger}_{\varphi} X \arrow[r, "\eta_{\coprod^{\dagger}_{\varphi} X}"] \arrow[dr, "f"'] & \mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X) \arrow[d, "f^{\sharp}"] & \coprod^{\dagger}_{\psi} X \arrow[l, "\xi_X"'] \arrow[dl, "(\xi_X)^{\mathbf{B}}(f)"] \\
 & B &
\end{tikzcd}$$

hence $(\theta^{\dagger\natural}_{\psi})_{X,B}((\xi_X)^{\mathbf{B}}(f))$ is a morphism from X to $\Delta^{\natural}_{\psi} B$. Now, for $s \in S$ and $x \in X_s$, we have that

$$\begin{aligned}
\big((\theta^{\dagger\natural}_{\psi})_{X,B}\big((\xi_X)^{\mathbf{B}}(f)\big)\big)_s(x) &= \big((\theta^{\dagger\natural}_{\psi})_{X,B}\big(f^{\sharp} \circ \xi_X\big)\big)_s(x) \\
&= \big((f^{\sharp} \circ \xi_X)_{\psi(s)_i}(x, s, \psi(s), i) \mid i \in |\psi(s)|\big) \\
&= \big(f^{\sharp}_{\psi(s)_i}\big((\xi_X)_{\psi(s)_i}(x, s, \psi(s), i)\big) \mid i \in |\psi(s)|\big) \\
&= f^{\sharp}_{\psi(s)}\big((\xi_X)_{\psi(s)_i}(x, s, \psi(s), i) \mid i \in |\psi(s)|\big) \\
&= f^{\sharp}_{\psi(s)}\Big((\xi_s)_i\big(v^{\varphi(s)_j}_j/(x, s, \varphi(s), j) \mid j \in |\varphi(s)|\big)_{i \in |\psi(s)|}\Big) \\
&= f^{\sharp}_{\psi(s)}\big(\xi_s^{\mathbf{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)}\big((x, s, \varphi(s), j) \mid j \in |\varphi(s)|\big)\big) \\
&= \xi^{\mathbf{B}}_s\big(f^{\sharp}_{\varphi(s)}\big((x, s, \varphi(s), j) \mid j \in |\varphi(s)|\big)\big)
\end{aligned}$$

because $\mathrm{Alg}_{\mathfrak{pd}}(\xi)$ is natural and $f^{\sharp}$ a morphism

$$\begin{aligned}
&= \xi^{\mathbf{B}}_s\big(f^{\sharp}_{\varphi(s)_j}(x, s, \varphi(s), j) \mid j \in |\varphi(s)|\big) \\
&= \xi^{\mathbf{B}}_s\big((\theta^{\dagger\natural}_{\varphi})_{X,B}(f)_s(x)\big) \\
&= \big((\xi^{\mathbf{B}})_X\big((\theta^{\dagger\natural}_{\varphi})_{X,B}(f)\big)\big)_s(x).
\end{aligned}$$

Therefore $(\xi^{\mathbf{B}})_X \circ (\theta^{\dagger\natural}_{\varphi})_{X,B} = (\theta^{\dagger\natural}_{\psi})_{X,B} \circ (\xi_X)^{\mathbf{B}}$, as asserted. $\square$

In the following proposition, which will be the basis to get the many-sorted term 2-institution of Fujiwara on **Set**, we construct a pseudo-functor from the 2-category $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category **Cat** and prove that there exists a pseudo-extranatural transformation from it to the functor from $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to **Cat** that is constantly **Set**.

Proposition 5.13. There exists a pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}_{\mathfrak{pd}}(\cdot)$ from the 2-category $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category **Cat**, obtained from the pseudo-functors $\mathrm{Alg}_{\mathfrak{pd}}$ and $\mathrm{Ter}_{\mathfrak{pd}}$, which sends a pair of signatures $(\mathbf{\Sigma}, \mathbf{\Lambda})$ to the category $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Lambda})$, and a pair of signature morphisms $(\mathbf{d}, \mathbf{e})$ from $(\mathbf{\Sigma}, \mathbf{\Lambda})$ to $(\mathbf{\Sigma}', \mathbf{\Lambda}')$ in $\mathbf{Sig}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to the functor $\mathbf{d}^{*}_{\mathfrak{pd}} \times \mathbf{d}^{\mathfrak{pd}}_{\diamond}$ from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma}') \times \mathbf{Ter}(\mathbf{\Lambda}')$. Furthermore, the family of functors $\mathrm{Tr} = (\mathrm{Tr}^{\mathbf{\Sigma}})_{\mathbf{\Sigma}\in\mathbf{Sig}_{\mathfrak{pd}}}$, together with the family $\theta = (\theta^{\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Sig}_{\mathfrak{pd}})}$, with $\theta^{\mathbf{d}}_{\mathbf{A},X} = \theta^{\dagger\natural}_{X,A}$, is a pseudo-extranatural transformation from the pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}_{\mathfrak{pd}}(\cdot)$ to the functor $\mathrm{K}_{\mathbf{Set}}$ from $\mathbf{Sig}_{\mathfrak{pd}}^{\mathrm{op}} \times \mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$ which picks $\mathbf{Set}$.

Proof. We restrict ourselves to prove that, for every transformation ξ in $\mathbf{Sig}_{\mathfrak{pd}}$ from the polyderivor $\mathbf{d}$ to the polyderivor $\mathbf{e}$, both from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, we have that the following equation holds

$$\theta^{\mathbf{e}} \circ (\mathrm{Tr}^{\mathbf{\Sigma}} * (\mathrm{Alg}_{\mathfrak{pd}}(\xi) \times 1)) = (\mathrm{Tr}^{\mathbf{\Lambda}} * (1 \times \mathrm{Ter}_{\mathfrak{pd}}(\xi))) \circ \theta^{\mathbf{d}}.$$

Let $f\colon \mathbf{A} \longrightarrow \mathbf{B}$ be a morphism in $\mathbf{Alg}(\mathbf{\Lambda})$ and $P\colon X \longrightarrow Y$ a morphism in $\mathbf{Ter}(\mathbf{\Sigma})$. Then we have the following configuration

$$\begin{tikzcd}
 & & (\mathbf{A},X) \arrow[r, "{(f,P)}"] \arrow[dll, mapsto] & (\mathbf{B},Y) \arrow[drr, mapsto] & & \\
(\mathbf{A},\coprod^{\dagger}_{\varphi} X) \arrow[r, "{(\mathbf{A},\xi_X)}"] \arrow[d, "{(f,\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P))}"] & (\mathbf{A},\coprod^{\dagger}_{\psi} X) \arrow[d, "{(f,\mathbf{e}^{\mathfrak{pd}}_{\diamond}(P))}"] & & & (\mathbf{d}^{*}_{\mathfrak{pd}}(\mathbf{A}),X) \arrow[r, "{(\xi^{\mathbf{A}},X)}"] \arrow[d, "{(f_{\varphi},P)}"] & (\mathbf{e}^{*}_{\mathfrak{pd}}(\mathbf{A}),X) \arrow[d, "{(f_{\psi},P)}"] \\
(\mathbf{B},\coprod^{\dagger}_{\varphi} Y) \arrow[r, "{(\mathbf{B},\xi_Y)}"'] & (\mathbf{B},\coprod^{\dagger}_{\psi} Y) \arrow[dd, mapsto] & & & (\mathbf{d}^{*}_{\mathfrak{pd}}(\mathbf{B}),Y) \arrow[r, "{(\xi^{\mathbf{B}},Y)}"'] \arrow[d, mapsto] & (\mathbf{e}^{*}_{\mathfrak{pd}}(\mathbf{B}),Y) \\
 & & (A_{\varphi})_X \arrow[rr, "(\xi^{\mathbf{A}})_X"] \arrow[dd, "P^{\mathbf{d}^{*}_{\mathfrak{pd}}(\mathbf{A})}"] & & (A_{\psi})_X \arrow[dd, "P^{\mathbf{e}^{*}_{\mathfrak{pd}}(\mathbf{A})}"] & \\
 & A_{\coprod^{\dagger}_{\varphi} X} \arrow[ur, "(\theta^{\dagger\natural}_{\varphi})_{X,A}"] \arrow[rr, "(\xi_X)^{\mathbf{A}}"] \arrow[dd, "\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)^{\mathbf{A}}"] & & A_{\coprod^{\dagger}_{\psi} X} \arrow[ur, "(\theta^{\dagger\natural}_{\psi})_{X,A}"] \arrow[dd, "\mathbf{e}^{\mathfrak{pd}}_{\diamond}(P)^{\mathbf{A}}"] & & \\
 & & (A_{\varphi})_Y \arrow[rr, "(\xi^{\mathbf{A}})_Y"] \arrow[dd, "(f_{\varphi})_Y"] & & (A_{\psi})_Y \arrow[dd, "(f_{\psi})_Y"] & \\
 & A_{\coprod^{\dagger}_{\varphi} Y} \arrow[ur, "(\theta^{\dagger\natural}_{\varphi})_{Y,A}"] \arrow[rr, "(\xi_Y)^{\mathbf{A}}"] \arrow[dd, "f_{\coprod^{\dagger}_{\varphi} Y}"] & & A_{\coprod^{\dagger}_{\psi} Y} \arrow[ur, "(\theta^{\dagger\natural}_{\psi})_{Y,A}"] \arrow[dd, "f_{\coprod^{\dagger}_{\psi} Y}"] & & \\
 & & (B_{\varphi})_Y \arrow[rr, "(\xi^{\mathbf{B}})_Y"] & & (B_{\psi})_Y & \\
 & B_{\coprod^{\dagger}_{\varphi} Y} \arrow[ur, "(\theta^{\dagger\natural}_{\varphi})_{Y,B}"] \arrow[rr, "(\xi_Y)^{\mathbf{B}}"] & & B_{\coprod^{\dagger}_{\psi} Y} \arrow[ur, "(\theta^{\dagger\natural}_{\psi})_{Y,B}"] & &
\end{tikzcd}$$

In the cube, the top, middle and bottom faces commute by the preceding lemma. The lateral faces commute by Lemma 4.27. The front face of the upper cube commutes by Proposition 5.10 and the front face of the lower cube commutes because f is a homomorphism. The back face of the top cube commutes because $\xi^{\mathbf{A}}$ is a homomorphism by Proposition 5.5, and the back face of the lower cube commutes by Proposition 5.6. □

From this proposition it follows immediately the following

Corollary 5.14. The quadruple $\mathfrak{Tm}_{\mathfrak{pd}} = (\mathbf{Sig}_{\mathfrak{pd}}, \mathrm{Alg}_{\mathfrak{pd}}, \mathrm{Ter}_{\mathfrak{pd}}, (\mathrm{Tr}, \theta))$ is a 2-institution on the category **Set**, the *many-sorted term* 2*-institution of Fujiwara*, or, simply, the *term* 2*-institution of Fujiwara*.

We close this section by pointing out that, for a Grothendieck universe $\boldsymbol{\mathcal{V}}$, such that $\boldsymbol{\mathcal{U}} \in \boldsymbol{\mathcal{V}}$, it is possible to prove that there exists an embedding from the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$ into the 2-category $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$, thus that the polyderivors together with the transformations between polyderivors constitute a concrete foundation for a bidimensional many-sorted algebra.

6. A 2-category of many-sorted specifications and the specification 2-institution of Fujiwara.

In this section we define a 2-category of specifications, $\mathbf{Spf}_{\mathfrak{pd}}$, with objects the specifications, morphisms from a specification into another the polyderivors between the underlying signatures of the specifications that are compatible with the equations, and 2-cells from a morphism into another a convenient class of transformations between the polyderivors.

For a polyderivor $\mathbf{d}\colon \Sigma \longrightarrow \Lambda$, the functor $\mathbf{d}^{\mathfrak{pd}}_{\diamond}$ of translation from $\mathbf{Ter}(\boldsymbol{\Sigma})$ to $\mathbf{Ter}(\boldsymbol{\Lambda})$ enables us to define the concept of $\mathfrak{pd}$-specification morphism from a specification into another.

Definition 6.1. Let $(\boldsymbol{\Sigma}, \mathcal{E})$ and $(\boldsymbol{\Lambda}, \mathcal{H})$ be specifications. An $\mathfrak{pd}$*-specification morphism from* $(\boldsymbol{\Sigma}, \mathcal{E})$ *to* $(\boldsymbol{\Lambda}, \mathcal{H})$ is a polyderivor $\mathbf{d}\colon \Sigma \longrightarrow \Lambda$ such that $(\mathbf{d}^{\mathfrak{pd}}_{\diamond})^2[\mathcal{E}] \subseteq \mathrm{Cn}_{\boldsymbol{\Lambda}}(\mathcal{H})$. We denote by $\mathbf{Spf}_{\mathfrak{pd}}$ the corresponding category.

Given two $\mathfrak{pd}$-specification morphisms $\mathbf{d}$ and $\mathbf{e}$ from $(\boldsymbol{\Sigma}, \mathcal{E})$ to $(\boldsymbol{\Lambda}, \mathcal{H})$, since $\mathbf{d}$ and $\mathbf{e}$ are, in particular, polyderivors from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$, we have, in principle, at our disposal all the transformations $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ from $\mathbf{d}$ to $\mathbf{e}$ as potential candidates for a concept of transformation between these $\mathfrak{pd}$-specification morphisms.

However, the condition of commutativity for the transformations between polyderivors is too much strict, because it requires, for every formal operation $\sigma\colon w \longrightarrow s$ in $\Sigma_{w,s}$, the strict equality $\xi_s \circ d(\sigma) = e(\sigma) \circ \xi_w$, and, actually, what could happen (and probably the most one reasonably can hope for), as was pointed out by Fujiwara in [Fuj60], is that, under the presence of equations, such a type of equation holds only modulus the congruence generated by the equations in the target specification. Therefore, for the $\mathfrak{pd}$-specification morphisms, the notion of transformation that we adopt, following the example of Fujiwara in [Fuj60], is that one where the strict equality between terms is replaced by the equality between them but relative to the congruence generated by the equations in the target specification. These transformations, in its turn, allow us to endow the category $\mathbf{Spf}_{\mathfrak{pd}}$ with a 2-category structure.

Definition 6.2. Let $\mathbf{d}$ and $\mathbf{e}\colon (\boldsymbol{\Sigma}, \mathcal{E}) \longrightarrow (\boldsymbol{\Lambda}, \mathcal{H})$ be $\mathfrak{pd}$-specification morphisms. A *transformation from* $\mathbf{d}$ *to* $\mathbf{e}$ is a choice function ξ for the family $(\mathrm{BT}_T(\Lambda)_{\varphi(s),\psi(s)})_{s \in S}$, such that, for every formal operation $\sigma\colon w \longrightarrow s$, it happens that $\xi_s \circ d(\sigma) \equiv_{\overline{\mathcal{H}}} e(\sigma) \circ \xi_w$.

Proposition 6.3. The specifications, the $\mathfrak{pd}$-specification morphisms, and the transformations between $\mathfrak{pd}$-specification morphisms determine a 2-category $\mathbf{Spf}_{\mathfrak{pd}}$.

The pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to **Cat** can be lifted up to the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$ as stated in the following

Proposition 6.4. There exists a pseudo-functor $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ from $\mathbf{Spf}_{\mathfrak{pd}}$ to **Cat** defined as follows

1. $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ sends a specification $(\boldsymbol{\Sigma}, \mathcal{E})$ to the category $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}(\boldsymbol{\Sigma}, \mathcal{E}) = \mathbf{Alg}(\boldsymbol{\Sigma}, \mathcal{E})$ of its models, i.e., the full subcategory of $\mathbf{Alg}(\boldsymbol{\Sigma})$ determined by those $\boldsymbol{\Sigma}$-algebras that satisfy all the equations in $\mathcal{E}$.
2. $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ sends a $\mathfrak{pd}$-specification morphism $\mathbf{d}$ from $(\boldsymbol{\Sigma}, \mathcal{E})$ to $(\boldsymbol{\Lambda}, \mathcal{H})$ to the functor $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{d}) = \mathbf{d}^{*}_{\mathfrak{pd}}$ from $\mathbf{Alg}(\boldsymbol{\Lambda}, \mathcal{H})$ to $\mathbf{Alg}(\boldsymbol{\Sigma}, \mathcal{E})$, obtained from the functor $\mathbf{d}^{*}_{\mathfrak{pd}}$ from $\mathbf{Alg}(\boldsymbol{\Lambda})$ to $\mathbf{Alg}(\boldsymbol{\Sigma})$ by bi-restriction.
3. $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ sends a transformation $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ from $\mathbf{d}$ to $\mathbf{e}$ to the natural transformation $\mathrm{Alg}_{\mathfrak{pd}}(\xi)$ from $\mathbf{d}^{*}_{\mathfrak{pd}}$ to $\mathbf{e}^{*}_{\mathfrak{pd}}$.

The pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to **Cat** can also be lifted to the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$ as stated in the following proposition.

Proposition 6.5. There exists a pseudo-functor $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}$ from $\mathbf{Spf}_{\mathfrak{pd}}$ to **Cat** defined as follows

1. $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}$ sends a specification $(\mathbf{\Sigma}, \mathcal{E})$ to the category $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{\Sigma}, \mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$, where $\mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$ is the quotient category $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$.
2. $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}$ sends a $\mathfrak{pd}$-specification morphism $\mathbf{d}$ from $(\mathbf{\Sigma}, \mathcal{E})$ to $(\mathbf{\Lambda}, \mathcal{H})$ to the functor $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{d})$ from the quotient category $\mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$ to the quotient category $\mathbf{Ter}(\mathbf{\Lambda}, \mathcal{H}) = \mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$, which assigns to a morphism $[P]_{\overline{\mathcal{E}}}$ from X to Y in $\mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$ the morphism $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{d})([P]_{\overline{\mathcal{E}}}) = [\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)]_{\overline{\mathcal{H}}} \colon \coprod^{\dagger}_{\varphi} X \longrightarrow \coprod^{\dagger}_{\varphi} Y$ in $\mathbf{Ter}(\mathbf{\Lambda}, \mathcal{H})$.
3. $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}$ sends a transformation $\xi \colon \mathbf{d} \rightsquigarrow \mathbf{e}$ from $\mathbf{d}$ to $\mathbf{e}$ to the natural transformation $\mathrm{Ter}_{\mathfrak{pd}}(\xi)$ from $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{d})$ to $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{e})$.

Furthermore, from the 2-category $\mathbf{Spf}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Spf}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$ there exists a pseudo-functor $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\cdot)$ and a pseudo-extranatural transformation $(\mathrm{Tr}^{\mathrm{sp}}, \theta^{\mathrm{sp}})$ from $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\cdot)$ to $\mathrm{K}_{\mathbf{Set}}$, and from this we get the following

Corollary 6.6. The quadruple $\mathfrak{Spf}_{\mathfrak{pd}} = (\mathbf{Spf}_{\mathfrak{pd}}, \mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}, \mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}, (\mathrm{Tr}^{\mathrm{sp}}, \theta^{\mathrm{sp}}))$ is a 2-institution on the category **Set**, the so-called *many-sorted specification* 2*-institution of Fujiwara*, or, simply, the *specification* 2*-institution of Fujiwara.*

From the pseudo-functor functor $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$, from $\mathbf{Spf}_{\mathfrak{pd}}$ to $\mathbf{Cat}$, to the pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}} \circ \mathrm{sig}^{\mathrm{op}}$, between the same 2-categories, there exists a pseudo-natural transformation, $\mathrm{In}_{\mathfrak{pd}}$, which sends a specification $(\mathbf{\Sigma}, \mathcal{E})$ to the full embedding $\mathrm{In}_{(\mathbf{\Sigma}, \mathcal{E})}$ of $\mathbf{Alg}(\mathbf{\Sigma}, \mathcal{E})$ into $\mathbf{Alg}(\mathbf{\Sigma})$. Besides, from the pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}} \circ \mathrm{sig}$, from $\mathbf{Spf}_{\mathfrak{pd}}$ to $\mathbf{Cat}$, to the pseudo-functor $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}$, between the same 2-categories, there exists a (strict) pseudo-natural transformation, $\mathrm{Pr}_{\mathfrak{pd}}$, given by the following data

1. For each specification $(\mathbf{\Sigma}, \mathcal{E})$, the projection functor $\mathrm{Pr}_{\overline{\mathcal{E}}}$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to the quotient category $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$.
2. For each specification morphism $\mathbf{d}$ from $(\mathbf{\Sigma}, \mathcal{E})$ to $(\mathbf{\Lambda}, \mathcal{H})$, the isomorphic identity natural transformation, denoted in this case by $\mathrm{Pr}_{\mathbf{d}}$, from the functor $\mathrm{Pr}_{\overline{\mathcal{H}}} \circ (\mathrm{Ter} \circ \mathrm{sig})(\mathbf{d})$, from the category $\mathbf{Ter}(\mathbf{\Sigma})$ to the category $\mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$, to the functor $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d}) \circ \mathrm{Pr}_{\overline{\mathcal{E}}}$, between the same categories.

Therefore we have obtained the following

Corollary 6.7. The pair $(\mathrm{sig}, (\mathrm{In}_{\mathfrak{pd}}, \mathrm{Pr}_{\mathfrak{pd}}))$ is a morphism of 2-institutions from the 2-institution $\mathfrak{Spf}_{\mathfrak{pd}}$ to the 2-institution $\mathfrak{Tm}_{\mathfrak{pd}}$.

7. Unification of domains.

Our main goal in this section is to prove that the theorem of Herbrand-Schmidt-Wang, about the reduction of heterogeneous first-order logic to homogeneous first-order logic, allows us to provide a natural example of a forward morphism between two institutions. To attain it we should begin by defining two institutions, the institution associated to the heterogeneous, or many-sorted, first-order logic and the institution associated to the homogeneous, or ordinary, first ordre logic. But before that, we choose, once and for all, a countably infinite set of variables $V = \{ v_n \mid n \in \mathbb{N} \}$.

The institution Ht, of *heterogeneous first-order logic*, is defined as follows. $\mathbf{Sig}^{\mathsf{Ht}}$, the category of *heterogeneous first-order signatures*, has as objects the quadruples $\mathcal{S} = (S, \Sigma, \Pi, j)$, where S is a set of sorts in $\boldsymbol{\mathcal{U}}$ such that $0 < \mathrm{card}(S) \leq \aleph_0$, Σ an S-sorted signature, Π an S-sorted predicate domain, i.e., an object of $\mathbf{Set}^{S^{\star}}$, and j a surjective mapping from V to S such that, for every $s \in S$, $j^{-1}[s]$, the fiber of j at s, is a countably infinite set; and as morphisms from $\mathcal{S} = (S, \Sigma, \Pi, j)$ to $\mathcal{S}' = (S', \Sigma', \Pi', j')$ the triples (ℓ, d, p), where $\ell \colon S \longrightarrow S'$ is a morphism in $\mathbf{Set}$, $d \colon \Sigma \longrightarrow \Sigma'_{\ell^{\star} \times \ell}$ a morphism in $\mathbf{Sig}(S)$, and $p \colon \Pi \longrightarrow \Pi'_{\ell^{\star}}$ a morphism in $\mathbf{Set}^{S^{\star}}$, such that ℓ satisfies the condition $\ell \circ j = j'$. Let us notice that from the equation $\ell \circ j = j'$ it follows that, for every $s \in S$, $j^{-1}[s]$ is included in $j'^{-1}[\ell(s)]$, and that ℓ is surjective. Our next goal is to define a contravariant functor $\mathrm{Mod}^{\mathsf{Ht}}$ from $\mathbf{Sig}^{\mathsf{Ht}}$ to $\mathbf{Cat}$. But before doing that we agree that, for an $\mathcal{S}$ in $\mathbf{Sig}^{\mathsf{Ht}}$, $\mathbf{Mod}^{\mathsf{Ht}}(\mathcal{S})$ denotes the category of *pointed heterogeneous* $\mathcal{S}$*-algebraic systems*, i.e., the category which has as objects the quadruples $\mathbf{A} = (A, F, R, a)$, where (A, F) is a $\mathbf{\Sigma} = (S, \Sigma)$-algebra such that, for each $s \in S$, $A_s \neq \varnothing$, R a mapping in $\mathbf{Set}^{S^{\star}}$ from Π to $\mathcal{R}_S(A) = (\mathrm{Sub}(A_w))_{w \in S^{\star}}$, and $a = (a_s)_{s \in S} \in \prod_{s \in S} A_s$; and as morphisms from $\mathbf{A} = (A, F, R, a)$ to $\mathbf{A}' = (A', F', R', a')$ those $\mathbf{\Sigma}$-homomorphisms f from (A, F) to

(A', F') such that, for each $w \in S^\star$, each $\pi \in \Pi_w$, and each $x \in A_w$, if $x \in R_\pi$, then $f_w(x) \in R'_\pi$, and, for each $s \in S$, $f_s(a_s) = a'_s$. Let us denote by $\mathrm{Mod}^{\mathsf{Ht}}$ the contravariant functor from $\mathbf{Sig}^{\mathsf{Ht}}$ to $\mathbf{Cat}$ which sends $\mathcal{S}$ in $\mathbf{Sig}^{\mathsf{Ht}}$ to $\mathbf{Mod}^{\mathsf{Ht}}(\mathcal{S})$ and a morphism (ℓ, d, p) from $\mathcal{S}$ to $\mathcal{S}'$ to the functor $\mathrm{Mod}^{\mathsf{Ht}}(\ell, d, p)$ from $\mathbf{Mod}^{\mathsf{Ht}}(\mathcal{S}')$ to $\mathbf{Mod}^{\mathsf{Ht}}(\mathcal{S})$ which assigns to $\mathbf{A}' = (A', F', R', a')$ precisely $(A'_\ell, F'_{\ell^\star \times \ell} \circ d, R'_{\ell^\star} \circ p, (a'_{\ell(s)})_{s \in S})$. To shorten notation, for each $(w, s) \in S^\star \times S$ and each $\sigma \in \Sigma_{w,s}$, we let $F'_{d(\sigma)}$ stand for the value of $F'_{\ell^\star \times \ell} \circ d$ at σ, and, for each $w \in S^\star$ and each $\pi \in \Pi_w$, we let $R'_{p(\pi)}$ stand for the value of $R'_{\ell^\star} \circ p$ at π. Our next goal is to define a functor $\mathrm{Sen}^{\mathsf{Ht}}$ from $\mathbf{Sig}^{\mathsf{Ht}}$ to $\mathbf{Cat}$. But before doing that we notice that, given a heterogeneous first-order signature $\mathcal{S} = (S, \Sigma, \Pi, j)$, for its underlying many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$, we have $\mathbf{T}_{\mathbf{\Sigma}}((j^{-1}[s])_{s \in S})$, the term $\mathbf{\Sigma}$-algebra on the S-sorted set $(j^{-1}[s])_{s \in S}$. Moreover, for the logical signature $\Lambda(\mathcal{S}) = (\Lambda_n(\mathcal{S}))_{n \in \mathbb{N}}$, where $\Lambda_1(\mathcal{S}) = \{\neg\} \cup \{\forall(v, s) \mid (v, s) \in \bigcup_{s \in S}(j^{-1}[s] \times \{s\})\}$, $\Lambda_2(\mathcal{S}) = \{\wedge, \vee, \rightarrow\}$, and $\Lambda_n(\mathcal{S}) = \varnothing$, if $n \neq 1, 2$, we have $\mathbf{Fm}^{\mathsf{Ht}}(\mathcal{S})$, the algebra of all heterogeneous $\mathcal{S}$-formulas, which is the free $\Lambda(\mathcal{S})$-algebra on $\mathrm{At}^{\mathsf{Ht}}(\mathcal{S}) = \bigcup_{(w,\pi) \in \coprod_{w \in S^\star} \Pi_w} (\{\pi\} \times \mathrm{T}_{\mathbf{\Sigma}}((j^{-1}[s])_{s \in S})_w)$, the set of all heterogeneous $\mathcal{S}$-atomic formulas. We agree that for $\mathcal{S}$ in $\mathbf{Sig}^{\mathsf{Ht}}$, $\mathbf{Sen}^{\mathsf{Ht}}(\mathcal{S})$ denotes the category which has as set of objects $\mathrm{Sen}^{\mathsf{Ht}}(\mathcal{S})$, the set of all heterogeneous $\mathcal{S}$-sentences, and as morphisms from φ to χ the triples $(\varphi, \mathcal{D}, \chi)$, where $\mathcal{D}$ is a proof of the sentence χ from the axioms together with φ. The composite of two morphisms $(\varphi, \mathcal{D}, \chi)$ and $(\chi, \mathcal{E}, \psi)$ is $(\varphi, \mathcal{E}\mathcal{D}, \psi)$, where $\mathcal{E}\mathcal{D}$ is the proof obtained by concatenating $\mathcal{D}$ and $\mathcal{E}$. Furthermore, for a sentence φ, the identity at φ is $(\varphi, \varnothing, \varphi)$. Then let the object mapping of $\mathrm{Sen}^{\mathsf{Ht}}$ be defined by sending $\mathcal{S}$ in $\mathbf{Sig}^{\mathsf{Ht}}$ to $\mathbf{Sen}^{\mathsf{Ht}}(\mathcal{S})$. Our next goal is to define the morphism mapping of $\mathrm{Sen}^{\mathsf{Ht}}$ which, for every morphism $(\ell, d, p)\colon \mathcal{S} \longrightarrow \mathcal{S}'$, must be a functor $\mathrm{Sen}^{\mathsf{Ht}}(\ell, d, p)\colon \mathbf{Sen}^{\mathsf{Ht}}(\mathcal{S}) \longrightarrow \mathbf{Sen}^{\mathsf{Ht}}(\mathcal{S}')$. Let (ℓ, d, p) be a morphism from $\mathcal{S}$ to $\mathcal{S}'$. Then, by applying the functor $\mathbf{d}^*$ from $\mathbf{Alg}(\mathbf{\Sigma}')$ to $\mathbf{Alg}(\mathbf{\Sigma})$, where $\mathbf{d} = (\ell, d)$, to the free $\mathbf{\Sigma}'$-algebra $\mathbf{T}_{\mathbf{\Sigma}'}((j'^{-1}[s'])_{s' \in S'})$, we obtain the $\mathbf{\Sigma}$-algebra $\mathbf{d}^*(\mathbf{T}_{\mathbf{\Sigma}'}((j'^{-1}[s'])_{s' \in S'}))$. Moreover, from the natural embedding $\mathrm{in}_{j,j'}$ of $(j^{-1}[s])_{s \in S}$ into $((j'^{-1}[s'])_{s' \in S'})_\ell = (j'^{-1}[\ell(s)])_{s \in S}$, we obtain, by the universal property of the free $\mathbf{\Sigma}$-algebra on an S-sorted set, the $\mathbf{\Sigma}$-homomorphism $\mathrm{in}^\sharp_{j,j'}$ from $\mathbf{T}_{\mathbf{\Sigma}}((j^{-1}[s])_{s \in S})$ to $\mathbf{d}^*(\mathbf{T}_{\mathbf{\Sigma}'}((j'^{-1}[s'])_{s' \in S'}))$. On the other hand, for every $w \in S^\star$ and every $\pi \in \Pi_w$, we have the mapping $t^{j,j'}_{w,\pi}$ from $(\mathrm{T}_{\mathbf{\Sigma}}((j^{-1}[s])_{s \in S})_w$ to $\mathrm{At}^{\mathsf{Ht}}(\mathcal{S}')$ which sends $(P_k)_{k \in |w|} \in (\mathrm{T}_{\mathbf{\Sigma}}((j^{-1}[s])_{s \in S})_w$ to $(p(\pi), ((\mathrm{in}^\sharp_{j,j'})_{w_k}(P_k))_{k \in |w|}) \in \mathrm{At}^{\mathsf{Ht}}(\mathcal{S}')$. From the family of mappings $t^{j,j'} = (t^{j,j'}_{w,\pi})_{w \in S^\star, \pi \in \Pi_w}$ we obtain, by the universal property of the coproduct, the mapping $\mathrm{At}^{\mathsf{Ht}}(t^{j,j'})$ from $\mathrm{At}^{\mathsf{Ht}}(\mathcal{S})$ to $\mathrm{At}^{\mathsf{Ht}}(\mathcal{S}')$. In addition, the S-sorted mapping $\mathrm{in}_{j,j'}$ determines a morphism $\overline{\mathrm{in}}_{j,j'}$ from $\Lambda(\mathcal{S})$ to $\Lambda(\mathcal{S}')$. Hence there is a forgetful functor from $\mathbf{Alg}(\Lambda(\mathcal{S}'))$ to $\mathbf{Alg}(\Lambda(\mathcal{S}))$. Let us denote by $\overline{\mathbf{in}}^*_{j,j'}(\mathbf{Fm}^{\mathsf{Ht}}(\mathcal{S}'))$ the value of the aforementioned functor at the $\Lambda(\mathcal{S}')$-algebra $\mathbf{Fm}^{\mathsf{Ht}}(\mathcal{S}')$. Then, by the universal property of the free $\Lambda(\mathcal{S})$-algebra on $\mathrm{At}^{\mathsf{Ht}}(\mathcal{S})$, there exists a unique $\Lambda(\mathcal{S})$-homomorphism $\mathrm{At}^{\mathsf{Ht}}(t^{j,j'})^\sharp$ from $\mathbf{Fm}^{\mathsf{Ht}}(\mathcal{S})$ to $\overline{\mathbf{in}}^*_{j,j'}(\mathbf{Fm}^{\mathsf{Ht}}(\mathcal{S}'))$ such that $\eta_{\mathrm{At}^{\mathsf{Ht}}(\mathcal{S}')} \circ \mathrm{At}^{\mathsf{Ht}}(t^{j,j'}) = \mathrm{At}^{\mathsf{Ht}}(t^{j,j'})^\sharp \circ \eta_{\mathrm{At}^{\mathsf{Ht}}(\mathcal{S})}$, where $\eta_{\mathrm{At}^{\mathsf{Ht}}(\mathcal{S})}$ is the canonical embedding of $\mathrm{At}^{\mathsf{Ht}}(\mathcal{S})$ into $\mathrm{Fm}^{\mathsf{Ht}}(\mathcal{S})$ and $\eta_{\mathrm{At}^{\mathsf{Ht}}(\mathcal{S}')}$ the canonical embedding of $\mathrm{At}^{\mathsf{Ht}}(\mathcal{S}')$ into $\overline{\mathrm{in}}^*_{j,j'}(\mathrm{Fm}^{\mathsf{Ht}}(\mathcal{S}'))$. Since the direct image of $\mathrm{Sen}^{\mathsf{Ht}}(\mathcal{S})$ under $\mathrm{At}^{\mathsf{Ht}}(t^{j,j'})^\sharp$ is included in $\mathrm{Sen}^{\mathsf{Ht}}(\mathcal{S}')$, we define the object mapping of the functor $\mathrm{Sen}^{\mathsf{Ht}}(\ell, d, p)$ from $\mathbf{Sen}^{\mathsf{Ht}}(\mathcal{S})$ to $\mathbf{Sen}^{\mathsf{Ht}}(\mathcal{S}')$ as the bi-restriction of $\mathrm{At}^{\mathsf{Ht}}(g^{j,j'})^\sharp$ to $\mathrm{Sen}^{\mathsf{Ht}}(\mathcal{S})$ and $\mathrm{Sen}^{\mathsf{Ht}}(\mathcal{S}')$. The morphism mapping of $\mathrm{Sen}^{\mathsf{Ht}}(\ell, d, p)$ is defined as follows. Given a morphism $\mathcal{D} = (\vartheta_k)_{k \in n}$ from φ to χ in $\mathbf{Sen}^{\mathsf{Ht}}(\mathcal{S})$, $\mathrm{Sen}^{\mathsf{Ht}}(\ell, d, p)(\mathcal{D})$ is $(\mathrm{Sen}^{\mathsf{Ht}}(\ell, d, p)(\varphi), (\mathrm{At}^{\mathsf{Ht}}(t^{j,j'})^\sharp(\vartheta_k))_{k \in n}, \mathrm{Sen}^{\mathsf{Ht}}(\ell, d, p)(\chi))$. Finally, let $\models^{\mathsf{Ht}}$ be the family $(\models^{\mathsf{Ht}}_{\mathcal{S}})_{\mathcal{S} \in \mathbf{Sig}^{\mathsf{Ht}}}$, where, for each $\mathcal{S} \in \mathrm{Sig}^{\mathsf{Ht}}$, $\models^{\mathsf{Ht}}_{\mathcal{S}}$ is the satisfaction relation associated to $\mathcal{S}$.

The institution Hm, of *homogeneous, or ordinary, first-order logic*, is defined as follows. $\mathbf{Sig}^{\mathsf{Hm}}$, the category of *homogeneous, or ordinary, first-order signatures*, has as objects the quadruples $\mathcal{S} = (S, \Sigma, \Pi, (\pi_s)_{s \in S})$, where S is a set in $\boldsymbol{\mathcal{U}}$ such that $0 < \mathrm{card}(S) \leq \aleph_0$, Σ an ordinary signature, i.e., an object of $\mathbf{Set}^{\mathbb{N}}$, Π an ordinary predicate domain, i.e., an object of $\mathbf{Set}^{\mathbb{N}}$, and $(\pi_s)_{s \in S}$ an injective mapping from S to Π_1; and as morphisms from $\mathcal{S} = (S, \Sigma, \Pi, (\pi_s)_{s \in S})$ to $\mathcal{S}' = (S', \Sigma', \Pi', (\pi'_{s'})_{s' \in S'})$ the triples (ℓ, d, p), where $\ell\colon S \longrightarrow S'$ is a surjective mapping, $d\colon \Sigma \longrightarrow \Sigma'$ a morphism in $\mathbf{Set}^{\mathbb{N}}$, and $p\colon \Pi \longrightarrow \Pi'$ a morphism in $\mathbf{Set}^{\mathbb{N}}$, such that, for every $s \in S$, $p_1(\pi_s) = \pi'_{\ell(s)}$. The remaining components of Hm are defined as usual, but taking into account that, for an homogeneous first-order signature $\mathcal{S} = (S, \Sigma, \Pi, (\pi_s)_{s \in S})$, an homogeneous $\mathcal{S}$-algebraic system $\mathbf{A}$ is a quadruple (A, F, R, a) consisting of an ordinary Σ-algebra (A, F), a mapping R in $\mathbf{Set}^{\mathbb{N}}$ from Π to $\mathcal{R}(A) = (\mathrm{Sub}(A^n))_{n \in \mathbb{N}}$, and a mapping a from S to A such that, for every $s \in S$, $a_s \in R_{\pi_s} (\subseteq A)$.

Next we define the functor DU, of *domain unification*, from $\mathbf{Sig}^{\mathsf{Ht}}$ to $\mathbf{Sig}^{\mathsf{Hm}}$. Let $\mathcal{S} = (S, \Sigma, \Pi, j)$ be a heterogeneous first-order signature. Then $\mathrm{DU}(\mathcal{S})$ is the homogeneous first-order signature $(S, \Sigma^{\mathrm{DU}}, \Pi^{\mathrm{DU}}, \mathrm{in}_S)$, where, for every $n \in \mathbb{N}$, $\Sigma^{\mathrm{DU}}_n = \coprod_{(w,s)\in S^n \times S} \Sigma_{w,s}$, $\Pi^{\mathrm{DU}}_n = \coprod_{w\in S^n} \Pi_w$, if $n \neq 1$, $\Pi^{\mathrm{DU}}_1 = (\coprod_{w\in S^1} \Pi_w) \coprod S$, and in_S the canonical embedding of S into Π^{DU}_1. The morphism mapping of DU is defined in the obvious way.

Following this we define, making use of the theorem of Herbrand-Schmidt-Wang, a natural transformation β from $\mathrm{Mod}^{\mathsf{Ht}}$ to $\mathrm{Mod}^{\mathsf{Hm}} \circ \mathrm{DU}^{\mathrm{op}}$. Let $\mathcal{S}$ be a heterogeneous first-order signature. Then $\beta_{\mathcal{S}}$ is the functor from $\mathbf{Mod}^{\mathsf{Ht}}(\mathcal{S})$ to $\mathbf{Mod}^{\mathsf{Hm}}(\mathrm{DU}(\mathcal{S}))$ which sends $\mathbf{A} = (A, F, R, a)$ to $(\coprod A, F^*, R^*, a^*)$, where, for every $(w, s) \in S^\star \times S$ and every $\sigma \in \Sigma_{w,s}$, $F^*_{((w,s),\sigma)}$ is the mapping from $(\coprod A)^{|w|}$ to $\coprod A$ which sends $(a_k, w_k)_{k\in|w|}$ in $(\coprod A)^{|w|}$ to $(F_\sigma((a_k)_{k\in|w|}), s)$ in $\coprod A$, if, for every $k \in |w|$, $a_k \in A_{w_k}$, and sends $(a_k, w_k)_{k\in|w|}$ to (a_s, s), otherwise; for every $w \in S^\star - S^1$ and every $\pi \in \Pi_w$, $R^*_{(\pi,w)}$ is the subset of $(\coprod A)^{|w|}$ defined as

$$\{x \in (\coprod A)^{|w|} \mid \exists r \in R_\pi(\forall k \in |w|(x_k = (r_k, w_k)))\},$$

for every $w \in S^\star$ such that $|w| = 1$ and every $\pi \in \Pi_w$, $R^*_{((\pi,w),0)}$ is defined as $R^*_{(\pi,w)}$, while, for every $s \in S$, $R^*_{(s,1)}$ is defined as $A_s \times \{s\}$; finally, a^* is $(a_s, s)_{s\in S}$. The definition of $\beta_{\mathcal{S}}$ on the morphisms is obvious.

In addition, the theorem of Herbrand-Schmidt-Wang also allows us to define a natural transformation α from $\mathrm{Sen}^{\mathsf{Ht}}$ to $\mathrm{Sen}^{\mathsf{Hm}} \circ \mathrm{DU}$. Let $\mathcal{S}$ be a heterogeneous first-order signature. Then $\alpha_{\mathcal{S}}$ is the functor from $\mathbf{Sen}^{\mathsf{Ht}}(\mathcal{S})$ to $\mathbf{Sen}^{\mathsf{Hm}}(\mathrm{DU}(\mathcal{S}))$ which sends an heterogeneous $\mathcal{S}$-sentence φ to the homogeneous $\mathrm{DU}(\mathcal{S})$-sentence $\alpha_{\mathcal{S}}(\varphi)$ obtained from φ by substituting simultaneously, for each expression of the form $\forall(v, s)\psi$ in φ an expression of the form $\forall v(\pi_s(v) \to \alpha_{\mathcal{S}}(\psi))$ (with the understanding that different variables in φ are replaced by different variables in $\alpha_{\mathcal{S}}(\varphi)$). The definition of $\alpha_{\mathcal{S}}$ on the morphisms is determined by the first part of the Herbrand-Schmidt-Wang theorem, bearing in mind that to the logical axioms it is added the set, denoted by $\Phi(\mathcal{S})$, of all of the following homogeneous $\mathrm{DU}(\mathcal{S})$-sentences:

1. $\exists v\, \pi_s(v)$, for every $s \in S$;
2. $\forall v_0, \ldots, v_{|w|-1}(\bigwedge_{k\in|w|} \pi_{w_k}(v_k) \to \pi_s(\sigma(v_0, \ldots, v_{|w|-1})))$, for every $(w, s) \in S^\star \times S$ and every $\sigma \in \Sigma_{w,s}$.

Also, by the theorem of Herbrand-Schmidt-Wang, we have that, for every heterogeneous first-order signature $\mathcal{S}$, every pointed heterogeneous $\mathcal{S}$-algebraic system $\mathbf{A}$, and every heterogeneous $\mathcal{S}$-sentence φ, $\mathbf{A} \models^{\mathsf{Ht}}_{\mathcal{S}} \varphi$ if and only if $\beta_{\mathcal{S}}(\mathbf{A}) \models^{\mathsf{Hm}}_{\mathrm{DU}(\mathcal{S})} \alpha_{\mathcal{S}}(\varphi)$. Moreover, $\beta_{\mathcal{S}}(\mathbf{A})$ is a model of the set $\Phi(\mathcal{S})$.

Therefore we have obtained a forward morphism $(\mathrm{DU}, \alpha, \beta)$ from the institution Ht to the institution Hm.

It seems appropriate to indicate that, related to the functor DU, the theorem of Herbrand-Schmidt-Wang obstructs the existence of a natural transformation from $\mathrm{Mod}^{\mathsf{Hm}} \circ \mathrm{DU}^{\mathrm{op}}$ to $\mathrm{Mod}^{\mathsf{Ht}}$. The essential reason for this is that it is not always possible to associate, in a natural way, a heterogeneous algebraic system to a homogeneous algebraic system. However, by the same theorem, for every heterogeneous first-order signature $\mathcal{S}$, it is possible to obtain a *partial* functor $\beta^{\S}_{\mathcal{S}}$ from $\mathbf{Mod}^{\mathsf{Hm}}(\mathrm{DU}(\mathcal{S}))$ to $\mathbf{Mod}^{\mathsf{Ht}}(\mathcal{S})$ in such a way that the family $\beta^{\S} = (\beta^{\S}_{\mathcal{S}})_{\mathcal{S}\in\mathbf{Sig}^{\mathsf{Ht}}}$ is natural in $\mathcal{S}$. Moreover, for each heterogeneous first-order signature $\mathcal{S}$, each model $\mathbf{B}$ of $\Phi(\mathcal{S})$, and each heterogeneous $\mathcal{S}$-sentence φ, $\beta^{\S}_{\mathcal{S}}(\mathbf{B}) \models^{\mathsf{Ht}}_{\mathcal{S}} \varphi$ if and only if $\mathbf{B} \models^{\mathsf{Hm}}_{\mathrm{DU}(\mathcal{S})} \alpha_{\mathcal{S}}(\varphi)$. For a fuller treatment of this topic we refer the reader to [ClA92] (we notice that the first author of this article also defined in [ClA92], inter alia, a certain type of 2-cell among institution morphisms from which he obtained a 2-category of institutions and a 2-functor from this 2-category to another 2-category, with 0-cells the categories of theories canonically associated to institutions).

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References

[Bar74] Barwise J (1974) *Axioms for abstract model theory.* Ann. Math. Logic **7**: 221–265

[Bén68] Bénabou J (1968) *Structures algebriques dans les categories.* Cahiers Topologie Géom. Différentielle **10**: 1–126

[BiL48] Birkhoff G, Lipson JD (1970) *Heterogeneous algebras.* J. Combinatorial Theory **8**: 115–133

[BoJ01] Borceux F, Janelidze G (2001) *Galois theories.* Cambridge University Press, UK

[ClA92] Climent J, Alcolea J (1992) *Instituciones y heterogeneidad.* Theoria **VII**: 65–85

[ClS05] Climent J, Soliveres J (2005) *The completeness theorem for monads in categories of sorted sets.* Houston J. Math. **31**: 103–129

[Coh81] Cohn P (1981) *Universal algebra.* D. Reidel Publishing Company, Dordrech: Holland/Boston: U.S.A./London: England

[Ehr65] Ehresmann Ch (1965) *Catégories et structures.* Dunod, France

[Fef74] Feferman S (1974) *Two notes on abstract model theory I. Properties invariant on the range of definable relations between structures.* Fund. Math. **82**: 153–164

[Fuj59] Fujiwara T (1959) *On mappings between algebraic systems.* Osaka Math. J. **11**: 153–172

[Fuj60] Fujiwara T (1960) *On mappings between algebraic systems, II.* Osaka Math. J. **12**: 253–268

[Göd33] Gödel K (1933) *Eine interpretation des intuitionistischen Aussagenkalküls*, Ergebnisse eines mathematischen Kolloquiums **4**: 39–40

[GoB84] Goguen J, Burstall R (1984) *Introducing institutions.* In: Clarke E (ed) *Proc. Logics of Programming Workshop.* Springer-Verlag, Berlin, pp 221–256

[GoB86] Goguen J, Burstall R (1986) *A study in the foundations of programming methodology: Specifications, institutions, charters and parchments.* In: Pitt D et alii. (eds) *Proc. Summer Workshop on Category Theory and Computer Programming*, Springer-Verlag, Berlin, pp 313–333

[GoM85] Goguen J, Meseguer J (1985) *Completeness of many-sorted equational logic.* Houston J. Math. **11**: 307–334

[GTW76] Goguen J, Thatcher J, Wagner E (1976) *An initial algebra approach to the specification, correctness, and implementation of abstract data types.* IBM Thomas J. Watson Research Center, Tecnical Report RC 6487

[Grä79] Grätzer G (1979) *Universal algebra, 2nd ed.* Springer-Verlag, New York-Heidelberg-Berlin

[Gra66] Gray J W (1966) *Fibred and cofibred categories.* In: Eilenberg S et alii. (eds) *Proc. Conf. Categorical Algebra*, Springer-Verlag, New York, pp 21–83

[Gro71] Grothendieck A (1971) *Catégories fibrées et descente (Exposé VI).* In: Grothendieck A (ed) Revêtements étales et groupe fondamental (SGA 1), Springer-Verlag, Berlin, pp 145–194.

[Her30] Herbrand J (1930) *Recherches sur la théorie de la démonstration*, Thesis at the University of Paris

[Hig63] Higgins P J (1963) *Algebras with a scheme of operators*, Math. Nachr. **27**: 115–132.

[Jac68] Jacobson N (1968) *Structure of rings.* Amer. Math. Soc. Colloq. Pub., vol 37, Providence, R.I., U.S.A.

[Kle65] Kleisli H (1965) *Every standard construction is induced by a pair of adjoint functors*, Proc. Amer. Math. Soc. **16**: 544–546

[Law63] Lawvere F W (1963) *Functorial semantics of algebraic theories*, Dissertation. Columbia University, U.S.A.

[Mac98] Mac Lane S (1998) *Categories for the working mathematician. 2nd ed.*, Springer-Verlag, New York

[Mat76] Matthiessen G (1976) *Theorie der Heterogenen Algebren*, Mathematik-Arbeits-Papiere, Nr. 3, Universität, Bremen, Germany

[Sch38] Schmidt A (1938) *Über deduktive Theorien mit mehreren Sorten von Grundingen*, Math. Ann. **115**: 485–506

[Sch61] Schmidt J (1961/62) *Algebraic operations and algebraic independence in algebras with infinitary operations*, Math. Japon. **6**: 77–112.

[TBG91] Tarlecki A, Burstall R, Goguen J (1991) *Some fundamental algebraic tools for the semantics of computation: Part 3. Indexed categories.* Theoret. Comput. Sci. **91**: 239–264

[TaV57] Tarski A, Vaught RL (1957) *Arithmetical extensions of relational systems*, Compositio Math. **13**: 81–102

[Wan52] Wang H (1952) *Logic of many-sorted theories*, J. Symbolic Logic **17**: 105–116