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On the relative power of polynomials with real, rational, and integer coefficients in proofs of termination of rewriting*

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Abstract

In the late seventies, Lankford and Dershowitz gave theoretical foundation to the use of polynomial interpretations with integer and real coefficients (respectively) in proofs of termination of rewriting. More than twenty five years after these pioneering works, however, the absence of true examples in the literature gave rise to some doubts about the possible benefits of using polynomials with real or rational coefficients. In this paper we prove that, in fact, there are rewriting systems which can be proved polynomially terminating by using polynomial interpretations with real coefficients whereas the proof cannot be achieved if polynomials only contain rational coefficients. We prove a similar statement regarding the use of rational coefficients versus integer coefficients.

Keywords: Polynomial orderings, program analysis, term rewriting, termination.

1 Introduction

Termination of rewriting is undecidable (even for TRSs containing only one rule [Dau92]) and lot of research has been devoted to develop methods and heuristics to achieve proofs of termination in restricted (and mechanizable) cases. Polynomial interpretations and the corresponding reduction orderings (pionereed by Lankford [Lan79]) are well-suited to achieve automatic or semiautomatic proofs of termination of rewriting [CL87, CMTU04, Gie95a, Lan79, Ste94a]. In Lankford's approach, symbols f are interpreted as polynomials $[f]$ with non-negative integer coefficients (with variables usually ranging on the set of positive integers). Terms t are inductively interpreted as polynomials $[t]$. Comparisons

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between terms t, s are achieved by comparing the value of $[t]$ and $[s]$ for all possible valuations of variables in t and s. The use of $\mathbb{N}$ as the interpretation domain guarantees the necessary well-foundedness of the induced ordering on terms. Termination of a TRS $\mathcal{R}$ is then ensured if, for each rewrite rule $l \to r$ of $\mathcal{R}$, $[l]$ is bigger than $[r]$.

Polynomials with *real* coefficients were proposed by Dershowitz [Der79] as an alternative to Lankford's polynomials over the naturals. Since $\mathbb{R}$ furnished with the usual ordering $>_{\mathbb{R}}$ is not well-founded, a *subterm* property (namely, $[f](x_1, \ldots, x_i, \ldots, x_k) >_{\mathbb{R}} x_i$ for all k-ary symbols f, $1 \leq i \leq k$, and $x_1, \ldots, x_k \in \mathbb{R}$) is explicitly required to ensure well-foundedness. Recently, a new framework for proving polynomial termination have been introduced were the need of the subterm property is overcomed. This is achieved by requiring that comparisons between terms are not below a given positive number δ: $[l] - [r] \geq \delta > 0$ [Luc05]. In Section 3 we give a more detailed account of these techniques.

Despite the fact that polynomials over the reals include polynomials over the naturals, no clear statement about the relative power of the derived termination techniques has been made so far. In fact, there are few examples in the literature showing the use of polynomials over the reals or rationals in proofs of termination. In [Der79], Dershowitz gave no concrete example of the use of such polynomial interpretations; no other such 'proper' examples seem to be ever handled by him [Der04]. Of course, after Dershowitz's foundational work, there have been attempts to use polynomial interpretations over the reals in practice, remarkably Steinbach's [Ste94a] and Giesl's [Gie95a] works. In [Ste94a], Steinbach gives a detailed description of a method to obtain polynomial interpretations over the reals which prove termination of a given TRS. However, no example in this paper involves either real or rational coefficients. In his PhD thesis, Steinbach considers the following TRS $\mathcal{R}$ [Ste94b, Example 5.82]:

```
f(X,h(Y)) -> g(f(X,Y),X)
i(X) -> f(X,X)
k(i(X),Y) -> k(k(X,Y),0)
```

and gives a polynomial proof of termination by using a polynomial interpretation with rational coefficients generated by his implementation:

$$
\begin{array}{rclcrcl}
[\mathtt{f}](x,y) & = & xy + 2.598x + 2502.619y & & [\mathtt{i}](x) & = & 26007.979x^2 + 1 \\
[\mathtt{g}](x,y) & = & x + y & & [\mathtt{k}](x,y) & = & xy + 1.26x + 10.079y \\
[\mathtt{h}](x) & = & 43.324x + 1 & & [\mathtt{0}] & = & 2
\end{array}
$$

However, termination of $\mathcal{R}$ can also be proved by using the following polynomial interpretation with only natural coefficients (automatically computed by MU-TERM [Luc04])!

```
[f](X1,X2) = X1.X2 + X1 + X2      [i](X) = X.X + 2.X + 1
[h](X) = X + 1                    [k](X1,X2) = X1 + X2
[g](X1,X2) = X1 + X2              [0] = 0
```

On the other hand, Giesl developed the system POLO which, in principle, is able to deal with polynomial interpretations over the reals [Gie95c]. On the basis of an (incomplete) version of Collins' algorithm to decide the satisfaction of a set of inequalities in the domain of real numbers [Col75], Giesl wrote that he *'knows of no TRS whose termination proof requires a polynomial interpretation with non-rational coefficients'* and, then, *'we have restricted the algorithm to rational instead of real numbers'* [Gie95a, Section 3]. Although this suggests the existence of some TRS requiring the use of rational coefficients (instead of natural coefficients) for achieving a proof of termination no such example is given in Giesl's papers and reports on POLO [Gie95a, Gie95b, Gie95c] or elsewhere. After more than 25 years, the following question remains open:

> Is there any TRS which can be proved polynomially terminating by using an ordering induced by polynomials with real (or rational) coefficients which *cannot* be proved polynomially terminating by using polynomials with (only) integer coefficients?

In this paper, we positively answer this question at two levels: for instance, the following TRS $\mathcal{R}_{\mathbb{R}}$:

```
f(f(X)) -> g(X)
g(c(X)) -> f(c(f(X)))
k(X,a2,b1) -> k(a1,X,b1)
k(a1',X,b1) -> k(X,a2',b1)
k(X,X,b1) -> k(g(X),b2,b2)
k(g(X),b3,b3) -> k(X,X,b4)
f(f(f(f(X)))) -> k(X,X,X)
```

is proved polynomially terminating by using the following polynomial interpretation with *real* coefficients:

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl} f(x) & = & \sqrt{2}\,x+1 & a_1 & = & 0 & b_1 & = & 1 \\ g(x) & = & 2x & a_2 & = & 1 & b_2 & = & 0 \\ k(x,y,z) & = & x+y+z & a_1' & = & 1 & b_3 & = & 1 \\ c(x) & = & x+5 & a_2' & = & 0 & b_4 & = & 0 \end{array}$$

whereas it cannot be proved terminating by using a polynomial intepretation with *rational* coefficients (this is proved below). In fact, in Section 4, we prove that there is an infinite number of such TRSs (from which the previous one is a particular case). We show how to build such TRSs and the corresponding polynomial proofs. Furthermore, the following TRS $\mathcal{R}_{\mathbb{Q}}$:

```
h(f(X)) -> g(X)
g(c(X)) -> h(c(f(X)))
k2(X,a2,b1) -> k2(a1,X,b1)
k2(a1',X,b1) -> k2(X,a2',b1)
k2(X,X,b1) -> k2(h(X),b2,b2)
k2(h(X),b3,b3) -> k2(X,X,b4)
k3(X,a2,a3,b1) -> k3(a1,X,a3,b1)
k3(a1',X,a3',b1) -> k3(X,a2',a3',b1)
k3(a1,X,a3,b1) -> k3(a1,a2,X,b1)
k3(a1',a2',X,b1) -> k3(a1',X,a3',b1)
k3(X,X,X,b1) -> k3(g(X),b2,b2,b2)
k3(g(X),b3,b3,b3) -> k3(X,X,X,b4)
f(f(f(X))) -> k2(X,X,X)
h(h(X)) -> k2(X,X,X)
f(f(f(f(X)))) -> k3(X,X,X,X)
h(h(X)) -> k3(X,X,X,X)
```

can be proved terminating by using a polynomial interpretation with *rational* coefficients whereas it cannot be proved terminating by using a polynomial intepretation with *natural* coefficients.

Apart from these results, which solve an old open problem, we believe than the proof method which has been used is also interesting and eventually useful or inspirating for other purposes. In both cases, the idea is to:

1. Use a TRS to establish a strong link between the values of the coefficients of the (linear) polynomial interpretations of some symbols. This is the case of the first two rules of $\mathcal{R}_{\mathbb{R}}$ above which establish that $f_1^2 = g_1$, where f_1 and g_1 are the coefficients of x in the (intended) linear interpretations for `f` and `g` above. Also, the first two rules of $\mathcal{R}_{\mathbb{Q}}$ establish that $h_1 f_1 = g_1$ (equivalently, $f_1 = \frac{g_1}{h_1}$).

2. Use TRSs to establish a concrete value for the coefficient of x in the (linear) polynomial interpretation of some (unary) symbols. This is the case of the rules defining `k` in $\mathcal{R}_{\mathbb{R}}$, which fix $g_1 = 2$, and also the case of the rules defining `k2` and `k3` in $\mathcal{R}_{\mathbb{Q}}$ which fix $h_1 = 2$ and $g_1 = 3$, respectively.

3. Finally, use some additional rule(s) to force linearity of all involved polynomials thus making the whole approach consistent.

2 Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ be the sets of natural, integer, rational and real numbers, respectively; given one of such sets N and $z \in N$, we let $N_z = \{x \in N \mid x \geq z\}$ and $N_{>z} = \{x \in N \mid x > z\}$.

A binary relation R on A is *terminating* (or well-founded) if there is no infinite sequence $a_1 \; R \; a_2 \; R \; a_3 \cdots$. A transitive and irreflexive relation $>$ on A is an ordering. Given $f : A^k \to A$ and $i \in \{1, \ldots, k\}$, we say that $>$ is monotonic on the i-th argument of f if, whenever $x > y$, we have $f(x_1, \ldots, x_{i-1}, x, \ldots, x_k) > f(x_1, \ldots, x_{i-1}, y, \ldots, x_k)$ for all $x, y, x_1, \ldots, x_k \in A$.

Throughout the paper, $\mathcal{X}$ denotes a countable set of variables and $\mathcal{F}$ denotes a signature, i.e., a set of function symbols $\{f, g, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

Rewrite systems and termination of rewriting

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ where R is a set of rewrite rules. A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ rewrites to s (at position p), written $t \xrightarrow{p}_{\mathcal{R}} s$ (or just $t \to s$), if $t|_p = \sigma(l)$ and $s = t[\sigma(r)]_p$, for some rule $\rho : l \to r \in R$, $p \in \mathcal{P}os(t)$ and substitution σ.

A TRS is terminating if $\to$ is terminating. The problem of proving termination of a TRS is equivalent to finding a well-founded, stable, and monotonic (strict) ordering $>$ on terms (i.e., a *reduction ordering*) which is *compatible* with the rules of the TRS, i.e., such that $l > r$ for all rules $l \to r$ of the TRS. Here, *monotonic* means that, for all k-ary symbol f and $i \in \{1, \ldots, k\}$, $>$ is monotonic on the i-th argument of f, when f is viewed as a mapping $f : \mathcal{T}(\mathcal{F}, \mathcal{X})^k \to \mathcal{T}(\mathcal{F}, \mathcal{X})$. *Stable* means that, whenever $t > s$, we have $\sigma(t) > \sigma(s)$ for all terms t, s and substitutions σ.

Term orderings can be obtained by giving appropriate interpretations to the function symbols of a signature. Given a signature $\mathcal{F}$, an $\mathcal{F}$-algebra is a pair $\mathcal{A} = (A, \mathcal{F}_A)$, where A is a set and $\mathcal{F}_A$ is a set of mappings $f_A : A^k \to A$ for each $f \in \mathcal{F}$ where $k = ar(f)$. We say that $\mathcal{A}$ is an $\mathcal{F}$-algebra over the reals (resp. rationals, integers, naturals) if $A \subseteq \mathbb{R}$ (resp. $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$). For a given valuation mapping $\alpha : \mathcal{X} \to A$, the evaluation mapping $[\alpha] : \mathcal{T}(\mathcal{F}, \mathcal{X}) \to A$ is inductively defined by $[\alpha](x) = \alpha(x)$ if $x \in \mathcal{X}$ and $[\alpha](f(t_1, \ldots, t_k)) = f_A([\alpha](t_1), \ldots, [\alpha](t_k))$ for $x \in \mathcal{X}$, $f \in \mathcal{F}$, $t_1, \ldots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Given a term t with $\mathcal{V}ar(t) = \{x_1, \ldots, x_n\}$, we write $[t]$ to denote the *function* $F_t : A^n \to A$ given by $F_t(a_1, \ldots, a_n) = [\alpha_{(a_1,\ldots,a_n)}](t)$ for each tuple $(a_1, \ldots, a_n) \in A^n$, where $\alpha_{(a_1,\ldots,a_n)}(x_i) = a_i$ for $1 \leq i \leq n$.

An *ordered* $\mathcal{F}$-algebra, is a triple $(A, \mathcal{F}_A, >_A)$, where $(A, \mathcal{F}_A)$ is a $\mathcal{F}$-algebra and $>_A$ is a (strict) ordering on A. Then, we can define an ordering $>$ on terms given by $t > s$ if and only $[\alpha](t) >_A [\alpha](s)$, for all $\alpha : \mathcal{X} \to A$. If $>_A$ is well-founded, then $>$ also is and the algebra is said to be well-founded [Zan03, Section 6.2.1].

3 Polynomial interpretations

Let $\mathbb{A}$ be a subring of a commutative ring $\mathbb{B}$. Let $x_1, \ldots, x_n \in \mathbb{B}$. For each n-tuple $(r_1, \ldots, r_n) = (r) \in \mathbb{N}^n$ (called a multi-index), we use vector notation, letting $(x) = (x_1, \ldots, x_n)$, and $\pi_r(x) = x_1^{r_1} \cdots x_n^{r_n}$. Such products π_r (for each multi-index $r \in \mathbb{N}^n$) are called (primitive) monomials in n variables over $\mathbb{B}$. $\sum_{i=1}^{n} r_i$ is the *degree* of the monomial; if $r_1 = r_2 = \cdots = r_n = 0$, then π_r is the constant monomial 1.

A polynomial P in n variables over $\mathbb{B}$ with coefficients in $\mathbb{A}$ is the sum $P = \sum_{r \in \mathbb{N}^n} a_r \cdot \pi_r(x)$, where $a_r \in \mathbb{A}$, of finitely many monomials in n variables over $\mathbb{B}$. The set of such polynomials is denoted by $\mathbb{A}[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are distinct variables. When variables $x_1, \ldots, x_n$ range on $\mathbb{B}$, P induces a (polynomial) function $P(x_1, \ldots, x_n) : \mathbb{B}^n \to \mathbb{B}$. In the following, we will write $\pi_r \in P$ (or just $\pi \in P$) to express that π_r is a monomial of a polynomial P such that $a_r \neq 0$. Moreover, for a given monomial $\pi_r \in P$ we let $\mathit{coef}(\pi_r) = a_r$, $\mathit{deg}_i(\pi_r) = r_i$ for $1 \leq i \leq n$, and $\mathcal{I}_\pi^+ = \{i \in \{1, \ldots, n\} \mid r_i \neq 0\}$. The degree $\mathit{deg}(P)$ of P is given by $\mathit{deg}(P) = \mathit{max}(\{\sum_{i \in \mathcal{I}_\pi^+} \mathit{deg}_i(\pi) \mid \pi \in P\})$.

In the following, we are mainly concerned with polynomials with real coefficients. Polynomial algebras $\mathcal{A} = (A, \mathcal{F}_A)$ for a given signature $\mathcal{F}$ consist of a carrier set $A \subseteq \mathbb{R}$ and a set of polynomials $[f]$ inducing mappings $[f] : A^k \to A$ which are collected in $\mathcal{F}_A$ for each k-ary symbol $f \in \mathcal{F}$.

3.1 Polynomial termination

According to Dershowitz's approach [Der79], we say that a TRS $(\mathcal{F}, R)$ is *D-polynomially compatible* if there is a polynomial $\mathcal{F}$-algebra $\mathcal{A} = (A, \mathcal{F}_A)$ such that, for all $l \to r \in R$, $[l] - [r] > 0$, i.e., the polynomial $P_{l,r} = [l] - [r]$ is positive for all possible valuations of variables in l and r. In order to guarantee termination of $\mathcal{R}$, we have to further ensure monotonicity of each polynomial function in the algebra and the subterm property: $\forall x_1, \ldots, x_k \in A$, and $i \in \{1, \ldots, k\}$, $[f](x_1, \ldots, x_{i-1}, x_i, \ldots, x_k) > x_i$. The subterm property is essential to guarantee well-foundedness of the underlying ordering. Polynomial algebras satisfying those monotonicity and subterm requirements are called here *D-polynomial algebras*. Then, if $\mathcal{R}$ is D-compatible with a D-polynomial algebra, then $\mathcal{R}$ is (D-polynomially) *terminating*.

Recently, a new framework for proving termination of TRSs by using polynomials over the reals has been introduced [Luc05]. Here, the carrier set A of the polynomial algebra $\mathcal{A} = (A, \mathcal{F}_A)$ must be *bounded from below* (i.e., there is $m \in \mathbb{R}$ such that $\forall x \in A, m \leq x$). We say that a TRS $(\mathcal{F}, R)$ is *L-polynomially compatible* if there is one of such algebras $\mathcal{A}$ and a positive number $\delta \in \mathbb{R}_{>0}$ such that, for all $l \to r \in R$, $[l] - [r] \geq \delta$. The role of δ in the previous formulation is to guarantee well-foundedness of the underlying ordering. Then, subterm property is not required anymore. The necessary monotonicity can be ensured (for all possible value of δ!) by requiring $\frac{\partial [f]}{\partial x_i} \geq 1$ for all symbol $f \in \mathcal{F}$ and $i \in \{1, \ldots, k\}$. Polynomial algebras $\mathcal{A} = (A, \mathcal{F}_A)$ satisfying this concrete monotonicity requirement are called here *L-polynomial algebras*. Then, if $\mathcal{R}$ is L-compatible with an L-polynomial algebra, then $\mathcal{R}$ is (L-polynomially) *terminating*.

In the following, when considering polynomial algebras, we will assume $A = \mathbb{R}_0$. This restriction does not affect our main results. Thus, disregarding the notion of (D or L) polynomial algebra which we have in mind, we always need to ensure positiveness of the polynomial $P_{l,r} = [l] - [r]$ for each rule $l \to r$ of the TRS. Thus, given a polynomial $\mathcal{F}$-algebra $\mathcal{A}$ for a given signature $\mathcal{F}$ (where we do not assume any special requirement for $\mathcal{A}$), we say that $\mathcal{A}$ is *compatible* with $\mathcal{R}$ if for all rule $l \to r$ in $\mathcal{R}$, $P_{l,r} > 0$.

By a linear polynomial algebra we mean a polynomial algebra whose polynomials are of the form $[f](x_1, \ldots, x_k) = a_k x_k + a_{k-1} x_{k-1} + \cdots + a_1 x_1 + a_0$ for all k-ary symbol $[f]$. We have the following result, which allows us to disregard from δ when dealing with linear L-polynomial algebras (see [Luc05] for a deeper discusion).

Proposition 1 *Let $\mathcal{R}$ be a TRS and $\mathcal{A}$ be a linear L-polynomial algebra. If $\mathcal{A}$ is compatible with $\mathcal{R}$, then $\mathcal{R}$ is L-polynomially compatible.*

PROOF. Consider an arbitrary rule $l \to r$ in $\mathcal{R}$. Since $\mathcal{A}$ is a linear polynomial interpretation, $P_{l,r} = [l] - [r]$ is also linear. Since $P_{l,r} > 0$, the linear polynomial $P_{l,r}$ cannot have negative coefficients for any rule in $\mathcal{R}$. Therefore, by [Luc05, Corollary 1], there is a positive δ which makes $\mathcal{R}$ L-polynomially compatible. □

We will extensively use the following obvious facts and assumptions: let P, Q be polynomials in one variable and $A \subseteq \mathbb{R}_0$:

Fact 1: If P is positive, then both the coefficient A_n of the monomial x^n of maximum degree and the constant coefficient A_0 are positive.

Fact 2: If $P \geq Q$ then $deg(P) \geq deg(Q)$.

Assumption 1: If $[f]$ is a linear polynomial interpreting a unary symbol f, then $deg([f]) = 1$ and the coefficient of maximum degree of $[f]$ is greather than or equal to 1; this is motivated by the requirement of subterm (in Dershowitz's framework) and monotonicity (in Lucas' framework) properties, i.e., this assumption actually holds when $D-$ or $L-$ linear polynomial algebras are considered.

4 Real vs. rational coefficients

Given $m \in \mathbb{N}_1$, consider the following TRS $\mathcal{S}_m$:

$\mathtt{f}^{m+1}\mathtt{(X) \text{ -> } g(X)}$
$\mathtt{g(c(X)) \text{ -> } f(c(f}^m\mathtt{(X)))}$

In the following results, we use f, g, and c (instead of $[f]$, $[g]$ and $[c]$ to denote the polynomials interpreting the unary symbols `f`, `g`, and `c`, respectively. Also, f_i (resp. g_i, c_i) is the coefficient of the monomial of degree i in the polynomial f interpreting a symbol `f`. Finally, n_f, n_g, and n_c are the degrees of such polynomials. We have the following:

Proposition 2 *Let $m \in \mathbb{N}_1$, $\mathcal{S}_m$ be as above and $\mathcal{A}$ be a polynomial algebra which is compatible with $\mathcal{S}_m$. Then, $n_f^{m+1} = n_g$.*

Proof. We have:

$$deg([\mathtt{f}^{m+1}\mathtt{(X)}]) = n_f^{m+1} \quad deg([\mathtt{g(X)}]) = n_g$$
$$deg([\mathtt{g(c(X))}]) = n_g n_c \quad deg([\mathtt{f(c(f}^m\mathtt{(X)))}]) = n_f n_c n_f^m = n_f^{m+1} n_c$$

Let $P_{l_1,r_1} = [\mathtt{f}^{m+1}\mathtt{(X)}] - [\mathtt{g(X)}]$. Let $n = deg(P_{l_1,r_1})$ and A_n be the coefficient of the monomial of degree n in P_{l_1,r_1}. Since $A_n \geq 0$, we must have $n_f^{m+1} \geq n_g$. Similarly, $n_g n_c \geq n_f^{m+1} n_c$. Since $n_c \geq 1$, we have $n_f^{m+1} \geq n_g \geq n_f^{m+1}$, i.e., $n_f^{m+1} = n_g$. □

Lemma 1 *Let f be a polynomial of degree n_f and $m \in \mathbb{N}_1$. Then, $f^m(x) = f_{n_f}^{\sum_{i=0}^{m-1} n_f^i} x^{n_f^m} + \cdots$.*

Proof. By induction on m. If $m = 1$, it is immediate. For the inductive case, consider

$$\begin{aligned}
f^{m+1}(x) &= f(f^m(x)) \\
&= f(f_{n_f}^{\sum_{i=0}^{m-1} n_f^i} x^{n_f^m} + \cdots) \\
&= f_{n_f}(f_{n_f}^{\sum_{i=0}^{m-1} n_f^i} x^{n_f^m} + \cdots)^{n_f} + \cdots \\
&= f_{n_f} f_{n_f}^{n_f \sum_{i=0}^{m-1} n_f^i} x^{n_f^{m+1}} + \cdots \\
&= f_{n_f} f_{n_f}^{\sum_{i=1}^{m} n_f^i} x^{n_f^{m+1}} + \cdots \\
&= f_{n_f}^{\sum_{i=0}^{m} n_f^i} x^{n_f^{m+1}} + \cdots
\end{aligned}$$

□

Proposition 3 *Let $m \in \mathbb{N}_1$, $\mathcal{S}_m$ be as above and $\mathcal{A}$ be a polynomial algebra which is compatible with $\mathcal{S}_m$. If $n_f = n_g = 1$, then, $f_1^{m+1} = g_1$. Moreover, either $n_c = 1$ or $f_1 = g_1 = 1$.*

PROOF. By Lemma 1, we have:

$$\begin{array}{rcl}
[\mathtt{f}^{m+1}\mathtt{(X)}] & = & f_{n_f}^{\sum_{i=0}^{m} n_f^i} x^{n_f^{m+1}} + \cdots \\
[\mathtt{g(X)}] & = & g_{n_g} x^{n_g} + \cdots \\
[\mathtt{g(c(X))}] & = & g_{n_g} c_{n_c}^{n_g} x^{n_g n_c} + \cdots \\
[\mathtt{f(c(f}^m\mathtt{(X)))}] & = & f_{n_f} c_{n_c}^{n_f} f_{n_f}^{n_f n_c \sum_{i=0}^{m-1} n_f^i} x^{n_f^{m+1} n_c} + \cdots \\
 & = & f_{n_f} c_{n_c}^{n_f} f_{n_f}^{n_c \sum_{i=1}^{m} n_f^i} x^{n_f^{m+1} n_c} + \cdots \\
 & = & f_{n_f}^{n_c \sum_{i=0}^{m} n_f^i} c_{n_c}^{n_f} x^{n_f^{m+1} n_c} + \cdots
\end{array}$$

Let $P_{l_1,r_1} = [\mathtt{f(f(X))}] - [\mathtt{g(X)}]$. Let $n = deg(P_{l_1,r_1})$ and A_n be the coefficient of the monomial of degree n in P_{l_1,r_1}. By Proposition 2, $A_n = f_{n_f}^{\sum_{i=0}^{m} n_f^i} - g_{n_g}$. Again, $A_n \geq 0$, i.e., $f_{n_f}^{\sum_{i=0}^{m} n_f^i} \geq g_{n_g}$. Similarly, $g_{n_g} c_{n_c}^{n_g} \geq f_{n_f}^{n_c \sum_{i=0}^{m} n_f^i} c_{n_c}^{n_f}$. Since $n_f = n_g = 1$ and $c_{n_c} > 0$, we actually have $f_1^{m+1} \geq g_1 \geq f_1^{(m+1)n_c}$ Since $n_f = 1$, we have $f_1 \geq 1$ (Assumption 1 above). Again, $n_c \geq 1$, and we have $f_1^{m+1} \geq g_1 \geq f_1^{(m+1)n_c} \geq f_1^{m+1}$, i.e., $f_1^{(m+1)n_c} = f_1^{m+1} = g_1$. This is possible only if either $n_c = 1$ or $f_1 = g_1 = 1$ hold. □

Proposition 3 gives us a first hint to induce an incompatibility of polynomial algebras over the rationals which could be overcome by polynomial algebras over the reals: for instance, if we can set $g_1 = 2$, then, since $f_1^{m+1} = g_1 = 2$, we could not prove polynomial termination by using polynomials over the rationals but this would be still possible with polynomials over the reals by letting $f_1 = \sqrt[m+1]{2}$. In the following, we show how to achieve this. Consider the following TRS $\mathcal{K}$:

```
k(X,a₂,b₁) -> k(a₁,X,b₁)        k(X,X,b₁) -> k(g(X),b₂,b₂)
k(a'₁,X,b₁) -> k(X,a'₂,b₁)      k(g(X),b₃,b₃) -> k(X,X,b₄)
```

In the following proposition we deal with multivariate monomials and polynomials. Given a multivariate polynomial f with variables $x_1, \ldots, x_n$, we let $n_{f,i}$ to be the maximum exponent of x_i in any monomial of f:

$$n_{f,i} = max(\{deg_i(\pi) \mid \pi \in f, i \in \mathcal{I}_\pi^+\})$$

Also, given $x_i, x_j \in \{x_1, \ldots, x_n\}$, for $i \neq j$, let $m_{f,i,j}$ be the maximum simultaneous degree of variables x_i, x_j in a monomial $\pi \in f$ such that $x_i, x_j \in \mathcal{I}_\pi^+$:

$$m_{f,i,j} = max(\{deg_i(\pi) + deg_j(\pi) \mid \pi \in f, \{i,j\} \subseteq \mathcal{I}_\pi^+\})$$

Note that $n_f \geq n_{f,i}$ for all $x_i \in \{x_1, \ldots, x_n\}$, and $n_{f,i} + n_{f,j} \geq m_{f,i,j}$. We have the following:

Proposition 4 *Let $\mathcal{K}$ be as above and $\mathcal{A}$ be a polynomial algebra which is compatible with $\mathcal{K}$. Then, $n_g \leq 2$.*

Proof. We have:

$$deg([\texttt{k(X,a}_2\texttt{,b}_1\texttt{)}]) = n_{k,1} \quad deg([\texttt{k(a}_1\texttt{,X,b}_1\texttt{)}]) = n_{k,2}$$
$$deg([\texttt{k(a}'_1\texttt{,X,b}_1\texttt{)}]) = n_{k,2} \quad deg([\texttt{k(X,a}'_2\texttt{,b}_1\texttt{)}]) = n_{k,1}$$

Thus, $n_{k,1} \geq n_{k,2}$ and $n_{k,2} \geq n_{k,1}$, i.e., $n_{k,1} = n_{k,2}$. Therefore, $n_{k,1} + n_{k,2} = 2n_{k,1} \geq m_{k,1,2}$. Now, since $n_{k,1} = n_{k,2}$, we have:

$$deg([\texttt{k(X,X,b}_1\texttt{)}]) = max(n_{k,1}, m_{k,1,2}) \quad deg([\texttt{k(g(X),b}_2\texttt{,b}_2\texttt{)}]) = n_{k,1}n_g$$
$$deg([\texttt{k(g(X),b}_3\texttt{,b}_3\texttt{)}]) = n_{k,1}n_g \quad deg([\texttt{k(X,X,b}_4\texttt{)}]) = max(n_{k,1}, m_{k,1,2})$$

Therefore, $max(n_{k,1}, m_{k,1,2}) = n_{k,1}n_g$. Now, we consider two cases:

1. If $n_{k,1} \geq m_{k,1,2}$, then $n_{k,1} = n_{k,1}n_g$ which is only possible if $n_g = 1 \leq 2$.
2. If $n_{k,1} < m_{k,1,2}$, then $m_{k,1,2} = n_{k,1}n_g$, i.e., $2n_{k,1} \geq n_{k,1}n_g$ which, since $n_{k,1} \geq 1$, is equivalent to $n_g \leq 2$.

□

Finally, consider the (one rule TRS) $\mathcal{L}_m$:

```
f^{2(m+1)}(X) -> k(X,X,X)
```

Then, we have:

Proposition 5 *Let $m \in \mathbb{N}_1$, $\mathcal{K}$ and $\mathcal{L}_m$ as above, $\mathcal{R} = \mathcal{K} \cup \mathcal{L}_m$ and $\mathcal{A}$ be a polynomial algebra which is compatible with $\mathcal{R}$. If $n_f = n_g = 1$, then, $n_k = 1$ and $g_1 = 2$.*

Proof. We have:

$$deg([\texttt{f}^{2(m+1)}\texttt{(X)}]) = n_f^{2(m+1)} \quad \text{and} \quad deg([\texttt{k(X,X,X)}]) = n_k$$

Therefore, since $n_f = 1$, and $n_f^{2(m+1)} = 1 \geq n_k$, we have $n_k = 1$. Now we know that $\mathcal{A}$ is a linear polynomial algebra. Then, we have:

$$[\texttt{k(X,a}_2\texttt{,b}_1\texttt{)}] = k_1x + k_2a_2 + k_3b_1 + k_0 \quad [\texttt{k(a}_1\texttt{,X,b}_1\texttt{)}] = k_2x + k_1a_1 + k_3b_1 + k_0$$
$$[\texttt{k(a}'_1\texttt{,X,b}_1\texttt{)}] = k_1a'_1 + k_2x + k_3b_1 + k_0 \quad [\texttt{k(X,a}'_2\texttt{,b}_1\texttt{)}] = k_1x + k_2a'_2 + k_3b_1 + k_0$$

Therefore, we conclude $k_1 = k_2$. Now, we have:

$$[\texttt{k(X,X,b}_1\texttt{)}] = 2k_1x + \cdots \quad [\texttt{k(g(X),b}_2\texttt{,b}_2\texttt{)}] = k_1g_1x + \cdots$$
$$[\texttt{k(g(X),b}_3\texttt{,b}_3\texttt{)}] = k_1g_1x + \cdots \quad [\texttt{k(X,X,b}_4\texttt{)}] = 2k_1x + \cdots$$

Then, $2k_1 = k_1g_1$. Since $k_1 \geq 1$ (Assumption 1), we have $g_1 = 2$. □

Corollary 1 *Let $m \in \mathbb{N}_1$, $\mathcal{S}_m$, $\mathcal{K}$ and $\mathcal{L}_m$ as above, $\mathcal{R} = \mathcal{S}_m \cup \mathcal{K} \cup \mathcal{L}_m$ and $\mathcal{A}$ be a polynomial algebra which is compatible with $\mathcal{R}$. Then, $\mathcal{A}$ is a linear polynomial algebra.*

PROOF. By Proposition 2, $n_f^{m+1} = n_g$ and by Proposition 4, $n_g \leq 2$. Thus, $n_f^{m+1} \leq 2$ and, since $m \geq 1$ and $n_f \in \mathbb{N}_1$, the only possibility is $n_f = n_g = 1$. By Proposition 5, $n_k = 1$ and $g_1 = 2$. Thus, by Proposition 3, $n_c = 1$, i.e., $\mathcal{A}$ is a linear polynomial algebra. □

Theorem 1 *Let $m \in \mathbb{N}_1$ and $\mathcal{R}_m = \mathcal{S}_m \cup \mathcal{K} \cup \mathcal{L}_m$, i.e., $\mathcal{R}_m$ is:*

$$\begin{array}{l}
\mathtt{f}^{m+1}(\mathtt{X}) \to \mathtt{g}(\mathtt{X}) \\
\mathtt{g}(\mathtt{c}(\mathtt{X})) \to \mathtt{f}(\mathtt{c}(\mathtt{f}^{m}(\mathtt{X}))) \\
\mathtt{k}(\mathtt{X},\mathtt{a}_2,\mathtt{b}_1) \to \mathtt{k}(\mathtt{a}_1,\mathtt{X},\mathtt{b}_1) \\
\mathtt{k}(\mathtt{a}'_1,\mathtt{X},\mathtt{b}_1) \to \mathtt{k}(\mathtt{X},\mathtt{a}'_2,\mathtt{b}_1) \\
\mathtt{k}(\mathtt{X},\mathtt{X},\mathtt{b}_1) \to \mathtt{k}(\mathtt{g}(\mathtt{X}),\mathtt{b}_2,\mathtt{b}_2) \\
\mathtt{k}(\mathtt{g}(\mathtt{X}),\mathtt{b}_3,\mathtt{b}_3) \to \mathtt{k}(\mathtt{X},\mathtt{X},\mathtt{b}_4) \\
\mathtt{f}^{2(m+1)}(\mathtt{X}) \to \mathtt{k}(\mathtt{X},\mathtt{X},\mathtt{X})
\end{array}$$

The following linear polynomial interpretation over the reals

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
f(x) & = & \sqrt[m+1]{2}\,x+1 & a_1 & = & 0 & b_1 & = & 1 \\
g(x) & = & 2x & a_2 & = & 1 & b_2 & = & 0 \\
k(x,y,z) & = & x+y+z & a'_1 & = & 1 & b_3 & = & 1 \\
c(x) & = & x+\left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil & a'_2 & = & 0 & b_4 & = & 0
\end{array}$$

proves termination of $\mathcal{R}_m$.

PROOF. See Appendix A. □

Corollary 2 *There is an infinite number of TRSs which can be proved polynomially terminating by using a polynomial algebra over the reals whereas they cannot be proved polynomially terminating by using a polynomial algebra over the rationals.*

PROOF. Theorem 1 shows that, for each $m \in \mathbb{N}_1$, the TRS $\mathcal{R}_m$ can be proved polynomially terminating by using a polynomial algebra over the reals. Assume that there is a polynomial algebra over the rationals which is compatible with $\mathcal{R}_m$ for some $m \in \mathbb{N}_1$. By Corollary 1, such an algebra must be linear. By Proposition 5, $g_1 = 2$ and by Proposition 3, $f_1^{m+1} = 2$ which is not possible if $f_1 \in \mathbb{Q}$. □

Interestingly, including $\mathcal{L}_m$ is essential to come into a true example of the power of polynomial algebras with real coefficients.

Example 1 *Termination of $\mathcal{R}'_1 = \mathcal{S}_1 \cup \mathcal{K}$ is proved by the following* simple *interpretation over the naturals:*

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
f(x) & = & 2x+1 & a_1 & = & 0 & b_1 & = & 2 \\
g(x) & = & 4x+1 & a_2 & = & 1 & b_2 & = & 0 \\
k(x,y,z) & = & xyz+x+3y+z & a'_1 & = & 1 & b_3 & = & 0 \\
c(x) & = & x+2 & a'_2 & = & 0 & b_4 & = & 0
\end{array}$$

5 Rational vs. natural coefficients

Consider now the TRS $\mathcal{S}$:

```
h(f(X)) -> g(X)
g(c(X)) -> h(c(f(X)))
```

We have the following:

Proposition 6 *Let $\mathcal{S}$ as above and $\mathcal{A}$ be a polynomial algebra which is compatible with $\mathcal{S}$. Then, $n_h n_f = n_g$.*

PROOF. We have:

$$\begin{array}{ll} deg([\texttt{h(f(X))}]) = n_h n_f & deg([\texttt{g(X)}]) = n_g \\ deg([\texttt{g(c(X))}]) = n_g n_c & deg([\texttt{h(c(f(X)))}]) = n_h n_c n_f \end{array}$$

By reasoning as in the proof of Proposition 2 and taking into account that $n_c \geq 1$ (Assumption 1), we conclude that $n_h n_f = n_g$. □

Proposition 7 *Let $\mathcal{S}$ as above and $\mathcal{A}$ be a polynomial algebra which is compatible with $\mathcal{S}$. If $n_h = n_g$, then $n_f = 1$ and $h_{n_h} f_1^{n_h} = g_{n_g}$. Moreover, either $n_c = 1$ or $f_1 = 1$.*

PROOF. By Proposition 6, $n_h n_f = n_g$. Since $n_h = n_g$, it must be $n_f = 1$. Now, consider:

$$\begin{array}{rclrcl} [\texttt{h(f(X))}] & = & h_{n_h} f_{n_f}^{n_h} x^{n_h n_f} + \cdots & [\texttt{g(X)}] & = & g_{n_g} x^{n_g} + \cdots \\ [\texttt{g(c(X))}] & = & g_{n_g} c_{n_c}^{n_g} x^{n_g n_c} + \cdots & [\texttt{h(c(f(X)))}] & = & h_{n_h} c_{n_c}^{n_h} f_{n_f}^{n_h n_c} x^{n_h n_f n_c} + \cdots \end{array}$$

Let $P_{l_1,r_1} = [\texttt{h(f(X))}] - [\texttt{g(X)}]$. Let $n = deg(P_{l_1,r_1})$ and A_n be the coefficient of the monomial of degree n in P_{l_1,r_1}. By Proposition 6, it must be $A_n = h_{n_h} f_{n_f}^{n_h} - g_{n_g}$. Again, $A_n \geq 0$, i.e., $h_{n_h} f_{n_f}^{n_h} \geq g_{n_g}$. Similarly, $g_{n_g} c_{n_c}^{n_g} \geq h_{n_h} c_{n_c}^{n_h} f_{n_f}^{n_h n_c}$. Since $n_h = n_g$ and $c_{n_c} > 0$, we actually have $h_{n_h} f_1^{n_h} \geq g_{n_g} \geq h_{n_h} f_1^{n_h n_c}$. Since $n_f = 1$, we must have $f_1 \geq 1$ (Assumption 1). Since (again) $n_c \geq 1$, we have $h_{n_h} f_1^{n_h} \geq g_{n_g} \geq h_{n_h} f_1^{n_h n_c} \geq h_{n_h} f_1^{n_h}$, i.e., $h_{n_h} f_1^{n_h n_c} = h_{n_h} f_1^{n_h} = g_{n_g}$. This entails that either $n_c = 1$ or $f_1 = 1$. □

Proposition 7 gives now a new hint to induce an incompatibility of polynomial algebras over the naturals which could be overcome by polynomial algebras over the rationals: for instance, if we can simultaneously set $n_h = n_g = 1$, $g_1 = 3$ and $h_1 = 2$, then, since $2f_1 = 3$, we could not prove polynomial termination by using polynomials over the rationals but this would be still possible with polynomials over the reals by letting $f_1 = \frac{3}{2}$. Now, given a unary symbol φ and $m \in \mathbb{N}$, consider the TRS $\mathcal{K}_{\varphi,m}$ consisting of the following $2(m-1)+2 = 2m$ rules defining the $m+1$-ary symbol k_m:

$$\begin{array}{lll} \texttt{k}_m\texttt{(X,a}_2\texttt{,a}_3\texttt{,...,a}_m\texttt{,b}_1\texttt{)} & \texttt{->} & \texttt{k}_m\texttt{(a}_1\texttt{,X,a}_3\texttt{,...,a}_m\texttt{,b}_1\texttt{)} \\ \texttt{k}_m\texttt{(a}'_1\texttt{,X,a}'_3\texttt{,...,a}'_m\texttt{,b}_1\texttt{)} & \texttt{->} & \texttt{k}_m\texttt{(X,a}'_2\texttt{,a}'_3\texttt{,...,a}'_m\texttt{,b}_1\texttt{)} \\ & \vdots & \end{array}$$

```
km(a1,a2,...,X,am,b1)     -> km(a1,a2,...,an-1,X,b1)
km(a'1,a'2,...,a'm-1,X,b1) -> km(a'1,a'2,...,X,a'm,b1)
km(X,...,X,b1)       -> km(φ(X),b2,...,b2)
km(φ(X),b3,...,b3)  -> km(X,...,X,b4)
```

We have the following:

Proposition 8 *Let $m \in \mathbb{N}$, $\varphi \in \mathcal{F}$ such that $ar(\varphi) = 1$, $\mathcal{K}_{\varphi,m}$ be as above, and $\mathcal{A}$ be a polynomial algebra which is compatible with $\mathcal{K}_{\varphi,m}$. Then, $n_\varphi \leq m$.*

PROOF. For each $i \in \{1, \ldots, m-1\}$, we have,

$$deg([\mathtt{k}_m(\mathtt{a}_1,\mathtt{a}_2,\ldots,\mathtt{X},\mathtt{a}_{i+1},\ldots,\mathtt{a}_m,\mathtt{b}_1)]) = n_{k_m,i}$$
$$deg([\mathtt{k}_m(\mathtt{a}_1,\mathtt{a}_2,\ldots,\mathtt{a}_i,\mathtt{X},\ldots,\mathtt{a}_m,\mathtt{b}_1)]) = n_{k_m,i+1}$$

and

$$deg([\mathtt{k}_m(\mathtt{a}'_1,\mathtt{a}'_2,\ldots,\mathtt{a}'_i,\mathtt{X},\ldots,\mathtt{a}'_m,\mathtt{b}_1)]) = n_{k_m,i+1}$$
$$deg([\mathtt{k}_m(\mathtt{a}'_1,\mathtt{a}'_2,\ldots,\mathtt{X},\mathtt{a}'_{i+1},\ldots,\mathtt{a}'_m,\mathtt{b}_1)]) = n_{k_m,i}$$

By reasoning as in the proofs of the previous propositions, we conclude that, For all $i \in \{1, \ldots, m-1\}$, that $n_{k_m,i} = n_{k_i,i+1}$. Therefore, $m\, n_{k_m,1} \geq m_{k_m,I}$ for any $I \subseteq \{1, \ldots, m\}$ such that I contains at least two elements: $|I| \geq 2$. Now, since $n_{k_m,i} = n_{k_i,i+1}$, we have:

$$deg([\mathtt{k}_m(\mathtt{X},\ldots,\mathtt{X},\mathtt{b}_1)]) = max(\{n_{k_m,1}\} \cup \{m_{k_m,I} \mid I \subseteq \{1,\ldots,m\}, |I| \geq 2\})$$
$$deg([\mathtt{k}_m(\varphi(\mathtt{X}),\mathtt{b}_2,\ldots,\mathtt{b}_2)]) = n_{k_m,1} n_\varphi$$

and

$$deg([\varphi(\mathtt{X}),\mathtt{b}_3,\ldots,\mathtt{b}_3)]) = n_{k_m,1} n_\varphi$$
$$deg([\mathtt{k}_m(\mathtt{X},\ldots,\mathtt{X},\mathtt{b}_4)]) = max(\{n_{k_m,1}\} \cup \{m_{k_m,I} \mid I \subseteq \{1,\ldots,m\}, |I| \geq 2\})$$

Therefore, $max(\{n_{k_m,1}\} \cup \{m_{k_m,I} \mid I \subseteq \{1,\ldots,m\}, |I| \geq 2\}) = n_{k_m,1} n_\varphi$. Now, we consider two cases:

1. If $n_{k_m,1}$ is the maximum, then $n_{k_m,1} = n_{k_m,1} n_\varphi$ which is only possible if $n_\varphi = 1$.
2. Otherwise, $m_{k_m,I} = n_{k_m,1} n_\varphi$ for some $I \subseteq \{1, \ldots, m\}, |I| \geq 2$. Then, $m\, n_{k_m,1} \geq n_{k_m,1} n_\varphi$ which, since $n_{k_m,1} \geq 1$, is equivalent to $m \geq n_\varphi$.

□

Finally, given $m \in \mathbb{N}_1$, we let $\mathcal{L}_m$:

$$\mathtt{f}^{m+1}(\mathtt{X}) \texttt{ -> } \mathtt{k}_m(\mathtt{X},\ldots,\mathtt{X},\mathtt{X})$$
$$\mathtt{h}^{\lceil \frac{m+1}{2} \rceil}(\mathtt{X}) \texttt{ -> } \mathtt{k}_m(\mathtt{X},\ldots,\mathtt{X},\mathtt{X})$$

Then, we have:

Proposition 9 *Let $m \in \mathbb{N}_1$, $\varphi \in \mathcal{F}$ such that $ar(\varphi) = 1$, $\mathcal{R} = \mathcal{K}_{\varphi,m} \cup \mathcal{L}_m$ be as above, and $\mathcal{A}$ be a polynomial algebra which is compatible with $\mathcal{R}$. If either $n_f = 1$ or $n_h = 1$, then, $n_{k_m} = 1$, $n_\varphi = 1$, and $\varphi_1 = m$.*

PROOF. We have:

$$deg([\mathtt{f}^{m+1}\mathtt{(X)}]) = n_f^{m+1} \quad \text{and} \quad deg([\mathtt{k}_m\mathtt{(X,\ldots,X,X)}]) = n_{k_m}$$
$$deg([\mathtt{h}^{\lceil\frac{m+1}{2}\rceil}\mathtt{(X)}]) = n_h^{\lceil\frac{m+1}{2}\rceil} \quad \text{and} \quad deg([\mathtt{k}_m\mathtt{(X,\ldots,X,X)}]) = n_{k_m}$$

Therefore, since $n_f = 1$ or $n_h = 1$, then, either $n_f^{m+1} = 1$ or $n_h^{\lceil\frac{m+1}{2}\rceil} = 1$. Thus, since $n_f^{m+1} \geq n_{k_m}$ and $n_h^{\lceil\frac{m+1}{2}\rceil} \geq n_{k_m}$, we have $n_{k_m} = 1$. Thus,

$$deg([\mathtt{k}_m\mathtt{(X,X,\ldots,X,b_1)}]) = 1 \text{ and } deg([\mathtt{k}_m\mathtt{(}\varphi\mathtt{(X),b_2,\ldots,b_2,b_2)}]) = n_\varphi$$

Since $1 \geq n_\varphi$, we conclude $n_\varphi = 1$. Now we know that $\mathcal{A}$ is a linear polynomial algebra. Then, for each $i \in \{1,\ldots,m-1\}$, we have,

$$[\mathtt{k}_m\mathtt{(a_1,a_2,\ldots,X,a_{i+1},\ldots,a_m,b_1)}] = k_{m,i}x + \cdots$$
$$[\mathtt{k}_m\mathtt{(a_1,a_2,\ldots,a_i,X,\ldots,a_m,b_1)}] = k_{m,i+1}x + \cdots$$

which implies $k_{m,i} \geq k_{m,i+1}$, and

$$[\mathtt{k}_m\mathtt{(a'_1,a'_2,\ldots,a'_i,X,\ldots,a'_m,b_1)}] = k_{m,i+1}x + \cdots$$
$$[\mathtt{k}_m\mathtt{(a'_1,a'_2,\ldots,X,a'_{i+1},\ldots,a'_m,b_1)}] = k_{m,i}x + \cdots$$

which implies $k_{m,i} \geq k_{m,i+1}$. Therefore, we conclude $k_{m,i} = k_{m,i+1}$ for all $i \in \{1,\ldots,m-1\}$. Now, we have:

$$[\mathtt{k}_m\mathtt{(X,\ldots,X,b_1)}] = mk_{m,1}x + \cdots \qquad [\mathtt{k}_m\mathtt{(}\varphi\mathtt{(X),b_2,\ldots,b_2)}] = k_1\varphi_1 x + \cdots$$
$$[\mathtt{k}_m\mathtt{(}\varphi\mathtt{(X),b_3,\ldots,b_3)}] = k_1\varphi_1 x + \cdots \qquad [\mathtt{k}_m\mathtt{(X,\ldots,X,b_4)}] = mk_1 x + \cdots$$

Then, we have: $mk_1 \geq k_1\varphi_1$ and $k_1\varphi_1 \geq mk_1$, i.e., $mk_1 = k_1\varphi_1$. Since $k_1 \geq 1$ (again, by monotonicity), we have $\varphi_1 = m$. □

Theorem 2 *Let $m \in \mathbb{N}_3$ be an odd natural number $\mathcal{R}_m = \mathcal{S} \cup \mathcal{K}_{\mathtt{g},m} \cup \mathcal{K}_{\mathtt{h},2} \cup \mathcal{L}'_m \cup \mathcal{L}'_2$, i.e., $\mathcal{R}_m$ is:*

```
h(f(X)) -> g(X)
g(c(X)) -> h(c(f(X)))
k₂(X,a₂,b₁) -> k₂(a₁,X,b₁)
k₂(a'₁,X,b₁) -> k₂(X,a'₂,b₁)
k₂(X,X,b₁) -> k₂(h(X),b₂,b₂)
k₂(h(X),b₃,b₃) -> k₂(X,X,b₄)
kₘ(X,a₂,a₃,...,aₘ,b₁)    -> kₘ(a₁,X,a₃,...,aₘ,b₁)
kₘ(a'₁,X,a'₃,...,a'ₘ,b₁)    -> kₘ(X,a'₂,a'₃,...,a'ₘ,b₁)
                              ⋮
kₘ(a₁,a₂,...,X,aₘ,b₁)    -> kₘ(a₁,a₂,...,aₘ₋₁,X,b₁)
kₘ(a'₁,a'₂,...,a'ₘ₋₁,X,b₁) -> kₘ(a'₁,a'₂,...,X,a'ₘ,b₁)
kₘ(X,...,X,b₁)       -> kₘ(g(X),b₂,...,b₂)
kₘ(g(X),b₃,...,b₃) -> kₘ(X,...,X,b₄)
f(f(f(X))) -> k₂(X,X,X)
```

```
h(h(X)) -> k₂(X,X,X)
f^{m+1}(X) -> km(X,...,X,X)
h^{⌈(m+1)/2⌉}(X) -> k_m(X,...,X,X)
```

Termination of $\mathcal{R}_m$ can be proved by using a linear polynomial interpretation over the rationals with domain $\mathbb{R}_1$ (or $\mathbb{Q}_1$):

$$\begin{array}{rclrcll} h(x) & = & 2x+1 & a_i & = & i & i \in \{1,\ldots,m\} \\ f(x) & = & \frac{m}{2}x & a'_i & = & m-i+1 & i \in \{1,\ldots,m\} \\ g(x) & = & mx & b_1 & = & m+1 & \\ c(x) & = & x+2 & b_i & = & 1 & i \in \{2,3,4\} \end{array}$$

$$\begin{array}{rcl} k_2(x,y,z) & = & x+y+z \\ k_m(x_1,\ldots,x_m,x_{m+1}) & = & x_1+x_2+\cdots+x_{m+1} \end{array}$$

Proof. See Appendix B. □

Theorem 2 shows that there is an infinite number of TRSs which can be proved polynomially terminating by using a linear polynomial algebra over the rationals. In fact, we can say more.

Corollary 3 *There is a TRS which can be proved polynomially terminating by using a polynomial interpretation with rational coefficients whereas it cannot be proved polynomially terminating by using a polynomial interpretation with integer coefficients.*

Proof. Theorem 2 shows that $\mathcal{R}_3$ can be proved polynomially terminating by using a polynomial algebra over the rationals. Assume that there is a polynomial algebra over the integers which is compatible with $\mathcal{R}_3$. By Proposition 8, $n_g \leq 3$ and $n_h \leq 2$. If $n_h = 1$, then by Proposition 9 (applied to $\mathcal{K}_{\mathtt{g},3} \cup \mathcal{L}'_3$), we have $n_g = 1$. Hence, by Proposition 6, which establishes $n_h n_f = n_g$, we have $n_f = 1$. If $n_h = 2$, then since $n_h n_f = n_g \leq 3$ (Proposition 6), it must be $n_f = 1$. However, this contradicts Proposition 9 (applied to $\mathcal{K}_{\mathtt{h},2} \cup \mathcal{L}'_2$), which imposes $n_h = 1$. Thus, the only possibility is $n_f = n_g = n_h = 1$. By Proposition 9, $n_{k_p} = 1$, $n_g = 1$, $g_1 = 3$, $n_{k_q} = 1$, $n_h = 1$, and $h_1 = 2$. Thus, by Proposition 7, $f_1 = \frac{g_1}{h_1} = \frac{3}{2} \notin \mathbb{Z}$. □

Note that $\mathcal{R}_3$ is the TRS $\mathcal{R}_{\mathbb{Q}}$ in the introduction. Although Corollary 3 is weaker than Corollary 2, we could naïvely argue that there is also an infinite number of TRSs which can be proved terminating with polynomial interpretations with rational coefficients, whereas they cannot be proved with polynomials with integer coefficients: just consider the disjoint union of $\mathcal{R}_3$ and any polynomially termiating TRS $\mathcal{R}$. However, the following result can be thought of as somehow equivalent 'in practice' to Corollary 2.

Corollary 4 *Let $m \in \mathbb{N}_3$ be an odd natural number and $\mathcal{R}_m = \mathcal{S} \cup \mathcal{K}_{\mathtt{g},m} \cup \mathcal{K}_{\mathtt{h},2} \cup \mathcal{L}'_m \cup \mathcal{L}'_2$. Then,*

1. *There is no linear polynomial algebra with integer coefficients which is compatible with $\mathcal{R}_m$.*

2. *Let $\mathcal{A}$ be a polynomial algebra with integer coefficients which is compatible with $\mathcal{R}_m$. Then, $n_h, n_f \geq 2$ and $4 \leq n_g \leq m$.*

PROOF.

1. Assume that $\mathcal{A}$ is a linear polynomial algebra over the naturals which is compatible with $\mathcal{R}_m$. Since $\mathcal{A}$ is linear, by Proposition 9 (applied to both $\mathcal{K}_{\mathbf{h},2} \cup \mathcal{L}'_2$ and $\mathcal{K}_{\mathbf{g},3} \cup \mathcal{L}'_3$), we have $h_1 = 2$ and $g_1 = m$. Thus, by linearity of $\mathcal{A}$ and by Proposition 7, $f_1 = \frac{g_1}{h_1} = \frac{m}{2} \notin \mathbb{Z}$. This leads to a contradiction.

2. By Proposition 8, in such an algebra, $n_h \leq 2$ and $n_g \leq m$. By Proposition 6, $n_h n_f = n_g$. If either $n_h = 1$ or $n_f = 1$, then by Proposition 6, we would get that $n_f = n_g = n_h = n_{k_2} = n_{k_m} = 1$ and $\mathcal{A}$ is, then, linear. However, this contradicts item 1 above. Therefore, it must be $n_h = 2$ and $n_f \geq 2$. By Proposition 6, $n_h n_f = n_g$, thus, $n_g \geq 4$.

□

Since automatic termination tools implementing polynomial termination with polynomial interpretations with integer coefficients (e.g., AProVE [GTSF04], C*i*ME [CMMU03], and TTT [HM05]) only use restricted versions of polynomial interpretations, the previous result is interesting in practice. Typically, such tools only generate linear, simple, and simple-mixed polynomial interpretations (according to Steinbach's classification, see [Ste94a, Ste94b]). In all of them, the maximum degree of a polynomial is 2. This means that no such tool could automatically prove termination of $\mathcal{R}_m$ for any odd $m \in \mathbb{N}_3$ because, according to Corollary 4, the generation of a polynomial of degree ≥ 4 would be necessary[1].

6 Conclusions

We have proven that there are TRSs which can be proved polynomially terminating by using polynomial interpretations with real coefficients whereas the proof cannot be achieved if polynomials only contain rational coefficients. We have also proven that there are TRSs which can be proved polynomially terminating by using polynomial interpretations with rational coefficients whereas the proof cannot be achieved if polynomials only contain integer coefficients.

Our results suggest that polynomial interpretations over the reals or rationals should be deeper considered as suitable tools for proving termination of rewriting. In [Luc05, Proposition 8], we already proved that such polynomial interpretations are strictly better than polynomial interpretations with integer coefficients when the generation of *non-monotonic* orderings is considered (as required, for instance, in the dependency pairs approach to termination of rewriting [AG00]). Now we have proven that this is also the case when monotonic orderings are necessary. Polynomials with real (or rational) coefficients, however, are not used in any of the most recent termination tools (e.g.,

[1] AProVE, however, permits to freely fix the maximum degree of polynomials and could be used to overcome this limitation.

APro VE, C*i*ME, and TTT) with the remarkable exception of MU-TERM. In fact, we used MU-TERM to find out the polynomial interpretation which proves termination of $\mathcal{R}_{\mathbb{Q}} = \mathcal{S}\cup\mathcal{K}_{\mathtt{h},2}\cup\mathcal{K}_{\mathtt{g},3}\cup\mathcal{L}'_2\cup\mathcal{L}'_3$ in the introduction. Although a direct proof of $\mathcal{R}_{\mathbb{Q}}$ was not possible, a proof of termination of $\mathcal{R}'_{\mathbb{Q}} = \mathcal{S}\cup\mathcal{K}_{\mathtt{h},2}\cup\mathcal{K}_{\mathtt{g},3}$ can be directly obtained by MU-TERM:

```
[h](X) = 2.X + 1                          [a1] = 1/2     [a3'] = 1/2
[f](X) = 3/2.X                            [a2] = 1       [b1] = 4
[g](X) = 3.X                              [a3] = 2       [b2] = 1
[c](X) = X + 2                            [a1'] = 2      [b3] = 1
[k2](X1,X2,X3) = X1 + X2 + X3             [a2'] = 1      [b4] = 1
[k3](X1,X2,X3,X4) = X1 + X2 + X3 + X4
```

The obtained polynomial interpretation is basically the same; the differences introduced by the automatic processing carried out by MU-TERM are not relevant in this setting. Moreover, although MU-TERM is not able to obtain it when $\mathcal{R}_{\mathbb{Q}}$ is loaded in the tool, it is easy to check that this polynomial interpretation also proves termination of $\mathcal{R}_{\mathbb{Q}}$.

As argued in [Luc05], the use of polynomials over the reals or rationals in proofs of termination is not far from the current procedures implemented in existing termination tools for dealing with polynomials with non-negative integer coefficients (see [Luc05, Remark 3]). We believe, however, that further research investigating appropriate implementation techniques is necessary.

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A Proof of Theorem 1

The following lemma is used in the proof.

Lemma 2 *Let $m \in \mathbb{N}_1$ and $f(x) = f_1 x + f_0$ be such that $f_1 \neq 1$. Then, $f^m(x) = f_1^m x + f_0 \frac{1-f_1^m}{1-f_1}$.*

Proof. By induction on m. If $m = 1$ it is obvious. For the inductive case, since $f^{m+1}(x) = f(f^m(x))$, by the Induction Hypothesis, we have:

$$\begin{array}{rclcl} f(f^m(x)) & = & f(f_1^m x + f_0 \frac{1-f_1^m}{1-f_1}) & = & f_1(f_1^m x + f_0 \frac{1-f_1^m}{1-f_1}) + f_0 \\ & = & f_1^{m+1} x + f_1 f_0 \frac{1-f_1^m}{1-f_1} + f_0 & = & f_1^{m+1} x + f_0(f_1 \frac{1-f_1^m}{1-f_1} + 1) \\ & = & f_1^{m+1} x + f_0(\frac{f_1 - f_1^{m+1}}{1-f_1} + 1) & = & f_1^{m+1} x + f_0(\frac{f_1 - f_1^{m+1} + 1 - f_1}{1-f_1}) \\ & = & f_1^{m+1} x + f_0(\frac{1-f_1^{m+1}}{1-f_1}) & & \end{array}$$

□

According to [Luc05], we have to find a positive δ such that (1) $>_\delta$ is monotonic and (2) $[l] - [r] \geq \delta$ for each rewrite rule of $\mathcal{R}_m$. Monotonicity easily follows (for any possible value of δ) because all coefficients accompanying all variables are greather than or equal to 1 (then, $\frac{\partial [f]}{\partial x_i} \geq 1$ holds for all symbols f and arguments $i \in \{1, \ldots, ar(f)\}$).

According to Proposition 1, we only need to further check compatibility of the rules, i.e., that, for all rule $l \to r \in \mathcal{R}_m$, $[l] - [r] > 0$.

1. Rule `f`$^{m+1}$`(X) -> g(X)`: By Lemma 2, we have

$$\begin{array}{rcl} [l_1] = [\mathtt{f}^{m+1}\mathtt{(X)}] & = & f_1^{m+1} x + f_0 \frac{1-f_1^{m+1}}{1-f_1} \\ & = & (\sqrt[m+1]{2})^{m+1} x + \frac{1-(\sqrt[m+1]{2})^{m+1}}{1-\sqrt[m+1]{2}} \\ & = & 2x - \frac{1}{1-\sqrt[m+1]{2}} \\ & = & 2x + \frac{1}{\sqrt[m+1]{2}-1} \\ [r_1] = [\mathtt{g(X)}] & = & 2x \end{array}$$

   Therefore, $P_{l_1,r_1} = [l_1] - [r_1] = \frac{1}{\sqrt[m+1]{2}-1} > 0$.

2. Rule `g(c(X)) -> f(c(f`^{m}`(X)))`: We have:

$$\begin{array}{rcl} [l_2] & = & [\mathtt{g(c(X))}] \\ & = & 2(x + \left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil) \\ & = & 2x + 2\left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil \end{array}$$

$$
\begin{array}{rcl}
[r_2] & = & [\mathtt{f(c(f}^m\mathtt{(X)))}] \\
& = & f_1(f_1^m x + f_0\frac{1-f_1^m}{1-f_1} + \left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil) + f_0 \\
& = & f_1^{m+1}x + f_1 f_0 \frac{1-f_1^m}{1-f_1} + f_1 \left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil + f_0 \\
& = & f_1^{m+1}x + f_0 \frac{f_1-f_1^{m+1}}{1-f_1} + f_1 \left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil + f_0 \\
& = & 2x + \frac{\sqrt[m+1]{2}-2}{1-\sqrt[m+1]{2}} + \sqrt[m+1]{2} \left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil + 1 \\
& = & 2x + \frac{\sqrt[m+1]{2}-2+1-\sqrt[m+1]{2}}{1-\sqrt[m+1]{2}} + \sqrt[m+1]{2} \left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil \\
& = & 2x + \frac{1}{\sqrt[m+1]{2}-1} + \sqrt[m+1]{2} \left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil
\end{array}
$$

Therefore, since $\frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \notin \mathbb{N}$ for any $m \in \mathbb{N}_1$, we have:

$$
\begin{array}{rcl}
P_{l_2,r_2} = [l_2] - [r_2] & = & (2-\sqrt[m+1]{2}) \left\lceil \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} \right\rceil - \frac{1}{\sqrt[m+1]{2}-1} \\
& > & (2-\sqrt[m+1]{2}) \frac{1}{(\sqrt[m+1]{2}-1)(2-\sqrt[m+1]{2})} - \frac{1}{\sqrt[m+1]{2}-1} \\
& = & \frac{1}{\sqrt[m+1]{2}-1} - \frac{1}{\sqrt[m+1]{2}-1} \\
& = & 0
\end{array}
$$

3. Rule `k(X,a`$_2$`,b`$_1$`) -> k(a`$_1$`,X,b`$_1$`)`: $[l_3] = [\mathtt{k(X,a_2,b_1)}] = x+2$ and $[r_3] = [\mathtt{k(a_1,X,b_1)}] = x+1$; thus, $P_{l_3,r_3} = 1 > 0$.
4. Rule `k(a`$'_1$`,X,b`$_1$`) -> k(X,a`$'_2$`,b`$_1$`)`: $[l_4] = [\mathtt{k(a'_1,X,b_1)}] = x+2$ and $[r_4] = [\mathtt{k(X,a'_2,b_1)}] = x+1$; thus, $P_{l_4,r_4} = 1 > 0$.
5. Rule `k(X,X,b`$_1$`) -> k(g(X),b`$_2$`,b`$_2$`)`: $[l_5] = [\mathtt{k(X,X,b_1)}] = 2x+1$ and $[r_5] = [\mathtt{k(g(X),b_2,b_2)}] = 2x$; thus, $P_{l_5,r_5} = 1 > 0$.
6. Rule `k(g(X),b`$_3$`,b`$_3$`) -> k(X,X,b`$_4$`)`: $[l_6] = [\mathtt{k(g(X),b_3,b_3)}] = 2x+2$ and $[r_6] = [\mathtt{k(X,X,b_4)}] = 2x$; thus, $P_{l_5,r_5} = 2 > 0$.
7. Rule `f`$^{2(m+1)}$`(X) -> k(X,X,X)`: By Lemma 2, $[l_6] = [\mathtt{f}^{2(m+1)}\mathtt{(X)}] = f_1^{2(m+1)}x + f_0\frac{1-f_1^{2(m+1)}}{1-f_1} = 4x + \frac{3}{\sqrt[m]{2}-1}$ and $[r_6] = [\mathtt{k(X,X,X)}] = 3x$; thus, $P_{l_6,r_6} = x + \frac{3}{\sqrt[m]{2}-1} > 0$ for all $x \geq 0$.

B Proof of Theorem 2

Monotonicity easily follows (for any possible value of δ) because all coefficients accompanying all variables are greather than or equal to 1. Now, according to Proposition 1, we only need to check compatibility of the rules, i.e., that, for all rule $l \to r \in \mathcal{R}_m$, $[l] - [r] > 0$.

1. Rule `h(f(X)) -> g(X)`: $[l_1] = [\mathtt{h(f(X))}] = 2\frac{m}{2}x + 1 = mx + 1$, $[r_1] = [\mathtt{g(X)}] = mx$. Therefore, $P_{l_1,r_1} = 1 > 0$.
2. Rule `g(c(X)) -> h(c(f(X)))`: $[l_2] = [\mathtt{g(c(X))}] = m(x+2) = mx + 2m$, $[r_2] = [\mathtt{h(c(f(X)))}] = 2(\frac{m}{2}x + 2) + 1 = mx + 5$. Therefore, $P_{l_2,r_2} = 2m - 5 > 0$ for all odd $m \in \mathbb{N}_3$.

3. Rules $\texttt{k}_m\texttt{(a}_1\texttt{,a}_2\texttt{,...,X,a}_{i+1}\texttt{,...,a}_m\texttt{,b}_1\texttt{)}$ `->` $\texttt{k}_m\texttt{(a}_1\texttt{,a}_2\texttt{,...,a}_i\texttt{,X,...,a}_m\texttt{,b}_1\texttt{)}$ for $i \in \{1, \ldots, m-1\}$:

$$\begin{array}{rcl} [l_3] & = & [\texttt{k}_m\texttt{(a}_1\texttt{,a}_2\texttt{,...,X,a}_{i+1}\texttt{,...,a}_m\texttt{,b}_1\texttt{)}] \\ & = & x + \sum_{j=1}^m a_j - a_i + b_1 \\ & = & x + \sum_{j=1}^m j - i + m + 1 \\ & = & x + \frac{m(m+1)}{2} - i + m + 1 \\ [r_3] & = & \texttt{k}_m\texttt{(a}_1\texttt{,a}_2\texttt{,...,a}_i\texttt{,X,...,a}_m\texttt{,b}_1\texttt{)} \\ & = & x + \sum_{j=1}^m a_j - a_{i+1} + b_1 \\ & = & x + \sum_{j=1}^m j - i - 1 + m + 1 \\ & = & x + \frac{m(m+1)}{2} - i + m \end{array}$$

Therefore, $P_{l_3,r_3} = [l_3] - [r_3] = 1 > 0$.

4. Rules $\texttt{k}_m\texttt{(a}'_1\texttt{,a}'_2\texttt{,...,a}'_i\texttt{,X,...,a}'_m\texttt{,b}_1\texttt{)}$ `->` $\texttt{k}_m\texttt{(a}'_1\texttt{,a}'_2\texttt{,...,X,a}'_{i+1}\texttt{,...,a}'_m\texttt{,b}_1\texttt{)}$ for $i \in \{1, \ldots, m-1\}$:

$$\begin{array}{rcl} [l_4] & = & [\texttt{k}_m\texttt{(a}'_1\texttt{,a}'_2\texttt{,...,a}'_i\texttt{,X,...,a}'_m\texttt{,b}_1\texttt{)}] \\ & = & x + \sum_{j=1}^m a'_j - a'_{i+1} + b_1 \\ & = & x + \sum_{j=1}^m (m-j+1) - (m-i-1+1) + m + 1 \\ & = & x + \sum_{j=1}^m j + i + 1 \\ & = & x + \frac{m(m+1)}{2} + i + 1 \\ [r_3] & = & \texttt{k}_m\texttt{(a}'_1\texttt{,a}'_2\texttt{,...,X,a}'_{i+1}\texttt{,...,a}'_m\texttt{,b}_1\texttt{))} \\ & = & x + \sum_{j=1}^m a'_j - a'_i + b_1 \\ & = & x + \sum_{j=1}^m (m-j+1) - (m-i+1) + m + 1 \\ & = & x + \sum_{j=1}^m j + i \\ & = & x + \frac{m(m+1)}{2} + i \end{array}$$

Therefore, $P_{l_4,r_4} = [l_4] - [r_4] = 1 > 0$.

5. Rules $\texttt{k}_m\texttt{(X,...,X,b}_1\texttt{)}$ `->` $\texttt{k}_m\texttt{(g(X),b}_2\texttt{,...,b}_2\texttt{)}$: $[l_5] = [\texttt{k}_m\texttt{(X,...,X,b}_1\texttt{)}] = mx + b_1 = mx + m + 1$ and $[r_5] = [\texttt{k}_m\texttt{(g(X),b}_2\texttt{,...,b}_2\texttt{)}] = g_1x + g_0 + mb_2 = mx + m$. Thus, $P_{l_5,r_5} = [l_5] - [r_5] = 1 > 0$.

6. Rules $\texttt{k}_m\texttt{(g(X),b}_3\texttt{,...,b}_3\texttt{)}$ `->` $\texttt{k}_m\texttt{(X,...,X,b}_4\texttt{)}$: $[l_6] = [\texttt{k}_m\texttt{(g(X),b}_3\texttt{,...,b}_3\texttt{)}] = g_1x + g_0 + mb_3 = mx + m$ and $[r_6] = [\texttt{k}_m\texttt{(X,...,X,b}_4\texttt{)}] = mx + b_4 = mx + 1$. Thus, $P_{l_6,r_6} = [l_6] - [r_6] = m - 1 > 0$ if $m \in \mathbb{N}_2$.

7. Rule `f(f(f(X)))` `->` $\texttt{k}_2\texttt{(X,X,X)}$: $[l_7] = [\texttt{f(f(f(X)))}] = \frac{m^3}{2^3}x$ and $[r_7] = [\texttt{k}_2\texttt{(X,X,X)}] = 3x$. Thus, $P_{l_7,r_7} = [l_7] - [r_7] = (\frac{m^3}{8} - 3)x = \frac{m^3-24}{8}x > 0$ for all $m \in \mathbb{N}_3$ and $x \geq 1$.

8. Rule `h(h(X))` `->` $\texttt{k}_2\texttt{(X,X,X)}$: $[l_8] = [\texttt{h(h(X))}] = 4x + 3$ and $[r_8] = [\texttt{k}_2\texttt{(X,X,X)}] = 3x$. Thus, $P_{l_8,r_8} = [l_8] - [r_8] = x + 3 > 0$ for all $x \geq 1$. Note that restricting the domain of the polynomial algebra to $\mathbb{R}_1$ instead to, e.g., $\mathbb{R}_0$ is essential to deal with this rule.

9. Rule $\texttt{f}^{m+1}\texttt{(X) -> km(X,...,X,X)}$: $[l_9] = [\texttt{f}^{m+1}\texttt{(X)}] = \frac{m^{m+1}}{2^{m+1}}x$ and $[r_9] = [\texttt{km(X,...,X,X)}] = (m+1)x$. Thus, $P_{l_9,r_9} = [l_9] - [r_9] = (\frac{m^{m+1}}{2^{m+1}} - (m+1))x = \frac{m^{m+1}-m-1}{2^{m+1}}x = \frac{m(m^m-1)-1}{2^{m+1}}x > 0$ for all $m \in \mathbb{N}_2$ and $x \geq 1$.

10. Rule $\texttt{h}^{\lceil \frac{m+1}{2} \rceil}\texttt{(X) -> k}_m\texttt{(X,...,X,X)}$: by Lemma 2,

$$
\begin{aligned}
[l_{10}] &= [\texttt{h}^{\lceil \frac{m+1}{2} \rceil}\texttt{(X)}] \\
&= 2^{\lceil \frac{m+1}{2} \rceil}x + \frac{1-2^{\lceil \frac{m+1}{2} \rceil}}{1-2} \\
&= 2^{\lceil \frac{m+1}{2} \rceil}x + 2^{\lceil \frac{m+1}{2} \rceil} - 1
\end{aligned}
$$

and $[r_{10}] = [\texttt{k}_m\texttt{(X,...,X,X)}] = (m+1)x$. Thus, $P_{l_{10},r_{10}} = [l_{10}] - [r_{10}] = (2^{\lceil \frac{m+1}{2} \rceil} - m - 1)x + 2^{\lceil \frac{m+1}{2} \rceil} - 1 > 0$ for all $m \in \mathbb{N}_2$ and $x \geq 1$.