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Informe Técnico / Technical Report

Ref. No.:	DSIC-II/07/06	Pages: 18
Title:	Dependency pairs for context-sensitive rewriting	
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Date:	March 6, 2006	
Keywords:	Dependency pairs, term rewriting, program analysis, termination.	

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Dependency pairs for context-sensitive rewriting*

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March 6, 2006

Abstract

Termination is one of the most interesting problems when dealing with context-sensitive rewrite systems. Although there is a good number of techniques for proving termination of context-sensitive rewriting (*CSR*), the dependency pair approach, one of the most powerful technique for proving termination of rewriting, has not been investigated in connection with proofs of termination of *CSR*. In this paper, we show how to use dependency pairs in proofs of termination of *CSR*. The implementation and practical use of the developed techniques yield a novel and powerful framework which dramatically improves the current state-of-the-art of methods for proving termination of *CSR*.

Keywords: Dependency pairs, term rewriting, program analysis, termination.

1 Introduction

A *replacement map* is a mapping $\mu : \mathcal{F} \rightarrow \mathcal{P}(\mathbb{N})$ satisfying $\mu(f) \subseteq \{1, \dots, k\}$, for each k -ary symbol f of a signature \mathcal{F} [Luc98]. We use them to discriminate the argument positions on which the rewriting steps are allowed. In this way, for a given Term Rewriting System (TRS), we obtain a restriction of rewriting which we call *context-sensitive rewriting* (*CSR* [Luc98, Luc02]). With *CSR* we can *achieve* a terminating behavior with non-terminating TRSs, by pruning (all) infinite rewrite sequences. Proving termination of *CSR* has been recently recognized as an interesting problem with several applications in the fields of term rewriting and programming languages (see [DLMMU04, GM04, Luc02, Luc06] for further motivation).

Several methods have been developed for proving termination of *CSR* under a replacement map μ for a given TRS \mathcal{R} (i.e., for proving the μ -*termination*

*This research was partly supported by Spanish MEC grant SELF TIN 2004-07943-C04-02.

of \mathcal{R}). In particular, a number of transformations which permit to treat termination of *CSR* as a standard termination problem have been described (see [GM04, Luc06] for recent surveys). Direct techniques like polynomial orderings and the context-sensitive version of the recursive path ordering have also been investigated [BLR02, GL02, Luc04b, Luc05]. Up to now, however, the *dependency pairs method* [AG00, HM04], one of the most powerful tools for proving termination of rewriting, has been not investigated in connection with proofs of termination of *CSR*. In this paper, we address this problem.

Roughly speaking, given a TRS \mathcal{R} , the dependency pairs associated to \mathcal{R} conform a new TRS $\text{DP}(\mathcal{R})$ which (together with \mathcal{R}) determines the so called *dependency chains* whose finiteness or infiniteness characterize termination or non-termination of \mathcal{R} . The dependency pairs can be presented as a *dependency graph*, where the absence of infinite chains can be analyzed by considering the *cycles* in the graph. In this paper, we show that these basic intuitions are also valid for *CSR*, although some important differences arise.

Example 1 Consider the following non-terminating TRS \mathcal{R} borrowing the well-known Toyama's example [GM99, Example 1]:

$$\begin{array}{l} c \rightarrow a \quad f(a, b, x) \rightarrow f(x, x, x) \\ c \rightarrow b \end{array}$$

together with $\mu(\mathbf{f}) = \{3\}$. As shown by Giesl and Middeldorp, among all existing transformations for proving termination of *CSR*, only the complete Giesl and Middeldorp's transformation [GM04] (yielding a TRS \mathcal{R}_C^μ) could be used in this case, but no concrete proof of termination for \mathcal{R}_C^μ is known yet. Furthermore, \mathcal{R}_C^μ has 13 dependency pairs and the dependency graph contains many cycles. In contrast to this, the *CS-TRS* has only one context-sensitive dependency pair and the corresponding dependency graph has no cycle! Thus, as we show below, a direct (and automatic) proof of μ -termination of \mathcal{R} is easy now.

Another interesting example is the following:

Example 2 Consider the non-terminating TRS \mathcal{R} from [Luc02, Section 11.1]:

```
terms(N) -> cons( recip(sqr(N)), terms(s(N)) )
sqr(0) -> 0
sqr(s(X)) -> s( add(sqr(X), dbl(X)) )
dbl(0) -> 0
dbl(s(X)) -> s( s(dbl(X)) )
add(0, X) -> X
add(s(X), Y) -> s( add(X, Y) )
first(0, X) -> nil
first(s(X), cons(Y, Z)) -> cons(Y, first(X, Z))
half(0) -> 0
half(s(0)) -> 0
half(s(s(X))) -> s( half(X) )
half(dbl(X)) -> X
```

which can be used to approximate the value of $\pi^2/6$. In particular, when $\mu(\mathbf{cons}) = \{1\}$ and $\mu(f) = \{1, \dots, ar(f)\}$ for all other symbols f , \mathcal{R} is μ -terminating. Hirokawa and Middeldorp report on a proof of termination of \mathcal{R}_{GM}^μ (which corresponds to the incomplete Giesl and Middeldorp’s transformation for proving termination of CSR, see [GM04]) by using the dependency pairs approach [HM05, Appendix A]. In particular, termination of \mathcal{R}_{GM}^μ implies the μ -termination of \mathcal{R} above. As remarked by Hirokawa and Middeldorp, there are 33 dependency pairs in \mathcal{R}_{GM}^μ . However, the context-sensitive dependency graph of \mathcal{R} consists of seven pairs with only four cycles, each of them containing a single dependency pair. Moreover, as explained below each of them can be removed by using the adaptation to CSR of Hirokawa and Middeldorp’s subterm criterion [HM04] which we develop below. Thus, the system is μ -terminating. This is in contrast with Hirokawa and Middeldorp’s proof which also use the subterm criterion (for $DP(\mathcal{R}_{GM}^\mu)$) but without being able to remove all cycles, thus requiring to do some constraint solving.

Although the technical development of the paper follows already existing results for unrestricted rewriting (mainly from [AG00, HM04]), a number of interesting theoretical issues are posed. Furthermore, as shown by our experiments, the obtained results dramatically improve the state-of-the-art of methods for proving termination of CSR.

After some preliminary definitions in Section 2, Section 3 introduces the general framework to compute and use context-sensitive dependency pairs for proving termination of CSR. Completeness is discussed in Section 4. Section 5 shows how to compute the (estimated) context-sensitive dependency graph. Section 6 adapts Hirokawa and Middeldorp’s subterm criterion to CSR. Section 7 provides an experimental evaluation of our techniques. Section 8 concludes.

2 Preliminaries

Throughout the paper, \mathcal{X} denotes a countable set of variables and \mathcal{F} denotes a signature, i.e., a set of function symbols $\{\mathbf{f}, \mathbf{g}, \dots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \rightarrow \mathbb{N}$. The set of terms built from \mathcal{F} and \mathcal{X} is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Positions p, q, \dots are represented by chains of positive natural numbers used to address subterms of t . Given positions p, q , we denote its concatenation as $p.q$. If p is a position, and Q is a set of positions, $p.Q = \{p.q \mid q \in Q\}$. We denote the empty chain by Λ . The set of positions of a term t is $Pos(t)$. The subterm at position p of t is denoted as $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced by s . We write $t \supseteq s$ if $s = t|_p$ for some $p \in Pos(t)$ and $t \triangleright s$ if $t \supseteq s$ and $t \neq s$. The symbol labelling the root of t is denoted as $root(t)$. A *context* is a term $C \in \mathcal{T}(\mathcal{F} \cup \{\square\}, \mathcal{X})$ with zero or more ‘holes’ \square (a fresh constant symbol).

A rewrite rule is an ordered pair (l, r) , written $l \rightarrow r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ where R is a set of rewrite

rules. Given $\mathcal{R} = (\mathcal{F}, R)$, we consider \mathcal{F} as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$, called *constructors* and symbols $f \in \mathcal{D}$, called *defined functions*, where $\mathcal{D} = \{\text{root}(l) \mid l \rightarrow r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$.

Context-sensitive rewriting. A mapping $\mu : \mathcal{F} \rightarrow \mathcal{P}(\mathbb{N})$ is a *replacement map* (or \mathcal{F} -map) if $\forall f \in \mathcal{F}, \mu(f) \subseteq \{1, \dots, \text{ar}(f)\}$ [Luc98]. Let $M_{\mathcal{F}}$ be the set of all \mathcal{F} -maps (or $M_{\mathcal{R}}$ for the \mathcal{F} -maps of a TRS (\mathcal{F}, R)). A binary relation R on terms is μ -monotonic if $t R s$ implies $f(t_1, \dots, t_{i-1}, t, \dots, t_k) R f(t_1, \dots, t_{i-1}, s, \dots, t_k)$ for every $t, s, t_1, \dots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. The set of μ -replacing positions $\mathcal{P}os^\mu(t)$ of $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is: $\mathcal{P}os^\mu(t) = \{\Lambda\}$, if $t \in \mathcal{X}$ and $\mathcal{P}os^\mu(t) = \{\Lambda\} \cup \bigcup_{i \in \mu(\text{root}(t))} i.\mathcal{P}os^\mu(t|_i)$, if $t \notin \mathcal{X}$. The set of replacing variables of t is $\mathcal{V}ar^\mu(t) = \{x \in \mathcal{V}ar(t) \mid \exists p \in \mathcal{P}os^\mu(t), t|_p = x\}$. The μ -replacing subterm relation \succeq_μ is given by $t \succeq_\mu s$ if there is $p \in \mathcal{P}os^\mu(t)$ such that $s = t|_p$. We write $t \triangleright_\mu s$ if $t \succeq_\mu s$ and $t \neq s$. In *context-sensitive rewriting* (CSR [Luc98]), we (only) contract *replacing* redexes: t μ -rewrites to s , written $t \hookrightarrow_\mu s$ (or $t \hookrightarrow_{\mathcal{R}, \mu} s$), if $t \xrightarrow{R} s$ and $p \in \mathcal{P}os^\mu(t)$. A TRS \mathcal{R} is μ -terminating if \hookrightarrow_μ is terminating. A term t is μ -terminating if there is no infinite μ -rewrite sequence $t = t_1 \hookrightarrow_\mu t_2 \hookrightarrow_\mu \dots \hookrightarrow_\mu t_n \hookrightarrow_\mu \dots$ starting from t . A pair (\mathcal{R}, μ) where \mathcal{R} is a TRS and $\mu \in M_{\mathcal{R}}$ is often called a CS-TRS.

Dependency pairs. Given a TRS $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ a new TRS $\text{DP}(\mathcal{R}) = (\mathcal{F}^\sharp, D(R))$ of *dependency pairs* for \mathcal{R} is given as follows: if $f(t_1, \dots, t_m) \rightarrow r \in R$ and $r = C[g(s_1, \dots, s_n)]$ for some defined symbol $g \in \mathcal{D}$ and $s_1, \dots, s_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, then $f^\sharp(t_1, \dots, t_m) \rightarrow g^\sharp(s_1, \dots, s_n) \in D(R)$, where f^\sharp and g^\sharp are new fresh symbols (called *tuple symbols*) associated to defined symbols f and g respectively [AG00]. Let \mathcal{D}^\sharp be the set of tuple symbols associated to symbols in \mathcal{D} and $\mathcal{F}^\sharp = \mathcal{F} \cup \mathcal{D}^\sharp$. As usual, for $t = f(t_1, \dots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write t^\sharp to denote the *marked term* $f^\sharp(t_1, \dots, t_k)$. Given $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$, let T^\sharp be the set $\{t^\sharp \mid t \in T\}$.

A reduction pair (\succeq, \sqsupset) consists of a stable and weakly monotonic quasi-ordering \succeq , and a stable and well-founded ordering \sqsupset satisfying either $\succeq \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \succeq \subseteq \sqsupset$. Note that *monotonicity is not required* for \sqsupset .

3 Context-Sensitive Dependency Pairs

Given a signature \mathcal{F} and $\mu \in M_{\mathcal{F}}$, let $\mathcal{T}_{\infty, \mu}$ be a set of minimal non- μ -terminating terms in the following sense: t belongs to $\mathcal{T}_{\infty, \mu}$ if t is non- μ -terminating and every strict subterm s (i.e., $t \triangleright s$) is μ -terminating. Note that every non- μ -terminating term s contains a minimal non- μ -terminating subterm $t \in \mathcal{T}_{\infty, \mu}$. Furthermore, it is obvious that $\text{root}(t) \in \mathcal{D}$ for all $t \in \mathcal{T}_{\infty, \mu}$.

Proposition 1 *Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. For all $t \in \mathcal{T}_{\infty, \mu}$, there exist $l \rightarrow r \in R$, a substitution σ , and terms u, s such that $r \succeq_\mu u \succeq s$, $\text{root}(s), \text{root}(u) \in \mathcal{D}$, $t \xrightarrow{\sigma}^{\epsilon} \sigma(l) \xrightarrow{\epsilon} \sigma(r) \succeq_\mu \sigma(u) \succeq \sigma(s)$, and $\sigma(s) \in \mathcal{T}_{\infty, \mu}$.*

PROOF. Consider an infinite μ -rewrite sequence starting from t . By definition of $\mathcal{T}_{\infty, \mu}$, all proper subterms of t are μ -terminating. Therefore, t has an inner reduction to an instance $\sigma(l)$ of the left-hand side of a rule $l \rightarrow r$ of \mathcal{R} : $t \xrightarrow{>\epsilon}^* \sigma(l) \xrightarrow{\epsilon} \sigma(r)$. Thus, we can write $t = f(t_1, \dots, t_k)$, $\sigma(l) = f(l_1, \dots, l_k)$ for some k -ary defined symbol f , and $t_i \xrightarrow{*} \sigma(l_i)$ for all i , $1 \leq i \leq k$. Since we assume all t_i to be μ -terminating for $1 \leq i \leq k$, the $\sigma(l_i)$ also are. Therefore, $\sigma(x)$ is μ -terminating for every $x \in \text{Var}(r) \subseteq \text{Var}(l)$. As $\sigma(r)$ is not μ -terminating, there is $t' \in \mathcal{T}_{\infty, \mu}$ such that $\sigma(r) \geq t'$. There must be a non-variable *replacing* subterm u of r (i.e., $r \geq_{\mu} u$ and $u \notin \mathcal{X}$) such that $\sigma(u) \geq t'$ and $\text{root}(u) \in \mathcal{D}$. Otherwise, we could write $\sigma(r) = C[r_1, \dots, r_n]$ for $C[\]$ being the *maximal replacing context* (see [Luc02]) of $\sigma(r)$ with $C[\]$ a *constructor context* (i.e., $C[\] \in \mathcal{T}(\mathcal{C} \cup \{\square\}, \mathcal{X})$), which implies that $\sigma(r)$ is μ -terminating. Note also that, because non- μ -terminating terms cannot occur in the substitution part, $t' = \sigma(s)$ for some non-variable subterm s of u ($u \geq s$) which, since $t' = \sigma(s) \in \mathcal{T}_{\infty, \mu}$, satisfies $\text{root}(u) \in \mathcal{D}$. \square

Implicit in the proof of the previous result is that s in Proposition 1 cannot be a strict subterm of l (see also [Der04]). Proposition 1 suggests to associate the following set $\overline{\text{DP}}(\mathcal{R}, \mu) \subseteq \text{DP}(\mathcal{R})$ of dependency pairs to a CS-TRS (\mathcal{R}, μ) :

Definition 1 Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let

$$\overline{\text{DP}}(\mathcal{R}, \mu) = \{l^{\sharp} \rightarrow s^{\sharp} \mid l \rightarrow r \in R, \exists u, r \geq_{\mu} u \geq s, \text{root}(u), \text{root}(s) \in \mathcal{D}, l \not\prec s\}$$

where we also define $\mu^{\sharp} \in M_{\mathcal{F}^{\sharp}}$ to be $\mu^{\sharp}(f) = \mu(f)$ if $f \in \mathcal{F}$ and $\mu^{\sharp}(f^{\sharp}) = \mu(f)$.

Note that, in the definition of $\overline{\text{DP}}(\mathcal{R}, \mu)$, we do not care of non-replacing subterms of the right-hand side r having no replacing occurrence of a defined symbol above.

Example 3 Consider the following TRS \mathcal{R} :

```
zeros -> 0:zeros
sel(0, X:XS) -> X
sel(s(N), X:XS) -> sel(N, XS)
```

with $\mu(\cdot) = \{1\}$ and $\mu(f) = \{1, \dots, \text{ar}(f)\}$ for all other symbols f . Then, $\overline{\text{DP}}(\mathcal{R}, \mu)$ is:

```
SEL(s(N), X:XS) -> SEL(N, XS)
```

and $\mu^{\sharp}(\text{SEL}) = \{1, 2\}$.

Given a CS-TRS $(\mathcal{P}, \mu^{\sharp})$ of dependency pairs associated to a CS-TRS (\mathcal{R}, μ) , an $(\mathcal{R}, \mathcal{P}, \mu^{\sharp})$ -chain is a sequence of pairs $u_i \rightarrow v_i \in \mathcal{P}$ such that there is a substitution σ satisfying $\sigma(v_i) \xrightarrow{*}_{\mathcal{R}, \mu^{\sharp}} \sigma(u_{i+1})$ for $i \geq 1$. Here, as usual we assume that different occurrences of dependency pairs do not share any variable.

Theorem 1 (Correctness) Let \mathcal{R} be a TRS and $\mu \in M_{\mathcal{R}}$. If there is no infinite $(\mathcal{R}, \overline{\text{DP}}(\mathcal{R}, \mu), \mu^{\sharp})$ -chain, then \mathcal{R} is μ -terminating.

PROOF. By contradiction. If \mathcal{R} is not μ -terminating, then there is $t \in \mathcal{T}_{\infty, \mu}$. By Proposition 1, and by Definition of $\overline{\text{DP}}(\mathcal{R}, \mu)$, there is a sequence

$$t^\sharp \xrightarrow{> \epsilon}^*_{\mathcal{R}, \mu^\sharp} \sigma(u_1) \xrightarrow{\epsilon}_{\overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp} \sigma(v_1) \xrightarrow{> \epsilon}^*_{\mathcal{R}, \mu^\sharp} \sigma(u_2) \xrightarrow{\epsilon}_{\overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp} \sigma(v_2) \cdots$$

where $\sigma(v_i) \in \mathcal{T}_{\infty, \mu}^\sharp$ for $i \geq 1$, from which we build an infinite $(\mathcal{R}, \overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp)$ -chain of dependency pairs, thus leading to a contradiction. \square

Example 4 Consider the TRS \mathcal{R} :

$$\begin{array}{l} \mathbf{g}(\mathbf{X}) \rightarrow \mathbf{h}(\mathbf{X}) \quad \mathbf{h}(\mathbf{d}) \rightarrow \mathbf{g}(\mathbf{c}) \\ \mathbf{c} \rightarrow \mathbf{d} \end{array}$$

together with $\mu(\mathbf{g}) = \mu(\mathbf{h}) = \emptyset$ [Zan97, Example 1]. $\overline{\text{DP}}(\mathcal{R}, \mu)$ is:

$$\begin{array}{l} \mathbf{G}(\mathbf{X}) \rightarrow \mathbf{H}(\mathbf{X}) \quad \mathbf{H}(\mathbf{d}) \rightarrow \mathbf{C} \\ \mathbf{H}(\mathbf{d}) \rightarrow \mathbf{G}(\mathbf{c}) \end{array}$$

Since $\mu^\sharp(\mathbf{G}) = \mu^\sharp(\mathbf{H}) = \emptyset$, no infinite $(\mathcal{R}, \overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp)$ -chain is possible. Thus, by Theorem 1, \mathcal{R} is μ -terminating.

Unfortunately, the definition of $\overline{\text{DP}}(\mathcal{R}, \mu)$ is *too weak* for capturing all cases of μ -termination, i.e., *completeness*, which is a very important property of the classical dependency pair approach, is *lost*.

Example 5 Consider the following CS-TRS (\mathcal{R}, μ) , where the rules of \mathcal{R} are:

$$\begin{array}{l} \mathbf{f}(\mathbf{a}) \rightarrow \mathbf{f}(\mathbf{f}(\mathbf{a})) \\ \mathbf{f}(\mathbf{a}) \rightarrow \mathbf{a} \end{array}$$

and $\mu(\mathbf{f}) = \emptyset$. This CS-TRS is clearly μ -terminating. However, $\overline{\text{DP}}(\mathcal{R}, \mu)$ is:

$$\begin{array}{l} \mathbf{F}(\mathbf{a}) \rightarrow \mathbf{F}(\mathbf{f}(\mathbf{a})) \\ \mathbf{F}(\mathbf{a}) \rightarrow \mathbf{F}(\mathbf{a}) \end{array}$$

with $\mu^\sharp(\mathbf{F}) = \emptyset$. The infinite chain

$$\mathbf{F}(\mathbf{a}) \xrightarrow{\epsilon}_{\overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp} \mathbf{F}(\mathbf{a}) \xrightarrow{\epsilon}_{\overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp} \cdots$$

of dependency pairs shows the lack of completeness of the previous approach.

In the following, we show that we can recover completeness by using a *different* set of context-sensitive dependency pairs.

4 Completeness of the CS-DP approach

Let $\mathcal{M}_{\infty, \mu}$ be a set of minimal non- μ -terminating terms in the following sense: t belongs to $\mathcal{M}_{\infty, \mu}$ if t is non- μ -terminating and every strict μ -replacing subterm s (i.e., $t \triangleright_\mu s$) is μ -terminating. Clearly, $\mathcal{T}_{\infty, \mu} \subseteq \mathcal{M}_{\infty, \mu}$. Obviously, if $t \in \mathcal{M}_{\infty, \mu}$, then $\text{root}(t)$ is a defined symbol. A rule $l \rightarrow r$ of a CS-TRS (\mathcal{R}, μ) is μ -conservative if $\mathcal{V}ar^\mu(r) \subseteq \mathcal{V}ar^\mu(l)$, i.e., all replacing variables in the right-hand-side are also replacing in the left-hand-side; (\mathcal{R}, μ) is μ -conservative if all its rules are (see [Luc06]).

Proposition 2 *Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. If \mathcal{R} is μ -conservative, then for all $t \in \mathcal{M}_{\infty, \mu}$, there exist $l \rightarrow r \in R$, a substitution σ and a term s such that $\text{root}(s) \in \mathcal{D}$, $r \succeq_{\mu} s$, $t \xrightarrow{>\epsilon} \sigma(l) \xrightarrow{\epsilon} \sigma(r) \succeq_{\mu} \sigma(s)$ and $\sigma(s) \in \mathcal{M}_{\infty, \mu}$.*

PROOF. Consider an infinite μ -rewrite sequence starting from t . By definition of $\mathcal{M}_{\infty, \mu}$, all proper μ -replacing subterms of t are μ -terminating. Therefore (since no μ -rewriting step is possible on non- μ -replacing subterms), t has an inner reduction to an instance $\sigma(l)$ of the left-hand side of a rule $l \rightarrow r$ of \mathcal{R} : $t \xrightarrow{>\epsilon} \sigma(l) \xrightarrow{\epsilon} \sigma(r)$. Thus, we can write $t = f(t_1, \dots, t_k)$, $\sigma(l) = f(l_1, \dots, l_k)$ for some k -ary defined symbol f , and $t_i \xrightarrow{*} \sigma(l_i)$ for all i , $1 \leq i \leq k$. Since all t_i are μ -terminating for $i \in \mu(f)$, the $\sigma(l_i)$ also are. By μ -conservativity of \mathcal{R} , it follows that $\sigma(x)$ is μ -terminating for every $x \in \text{Var}^{\mu}(r) \subseteq \text{Var}^{\mu}(l)$. Since $\sigma(r)$ is non- μ -terminating, it contains a replacing subterm $t' \in \mathcal{M}_{\infty, \mu}$: $\sigma(r) \succeq_{\mu} t'$. Since variables $x \in \text{Var}^{\mu}(r)$ satisfy that $\sigma(x)$ is μ -terminating, it follows that there is a nonvariable subterm s of r (i.e., $r \succeq_{\mu} s$) such that $\sigma(s) = t'$ as required. Thus, $\text{root}(s) \in \mathcal{D}$. \square

Definition 2 *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let*

$$\text{DP}(\mathcal{R}, \mu) = \{l^{\sharp} \rightarrow s^{\sharp} \mid l \rightarrow r \in R, r \succeq_{\mu} s, \text{root}(s) \in \mathcal{D}, l \not\prec_{\mu} s\}$$

where $\mu^{\sharp}(f) = \mu(f)$ if $f \in \mathcal{F}$ and $\mu^{\sharp}(f^{\sharp}) = \mu(f)$ if $f \in \mathcal{D}$.

Clearly, $\text{DP}(\mathcal{R}, \mu) \subseteq \overline{\text{DP}}(\mathcal{R}, \mu)$. We have the following.

Theorem 2 (Completeness) *Let \mathcal{R} be a TRS and $\mu \in M_{\mathcal{R}}$. If \mathcal{R} is μ -terminating, then there is no infinite $(\mathcal{R}, \text{DP}(\mathcal{R}, \mu), \mu^{\sharp})$ -chain.*

PROOF. By contradiction. If there is an infinite $(\mathcal{R}, \text{DP}(\mathcal{R}, \mu), \mu^{\sharp})$ -chain, then there is a substitution σ and dependency pairs $u_i \rightarrow v_i \in \text{DP}(\mathcal{R}, \mu)$ for $i \geq 1$ such that

$$\sigma(u_1) \xrightarrow{\epsilon}_{\text{DP}(\mathcal{R}, \mu), \mu^{\sharp}} \sigma(v_1) \xrightarrow{>\epsilon}_{\mathcal{R}, \mu^{\sharp}} \sigma(u_2) \xrightarrow{\epsilon}_{\text{DP}(\mathcal{R}, \mu), \mu^{\sharp}} \sigma(v_2) \cdots$$

Assume that $u_i = f_i^{\sharp}(\overline{u}_i)$ and $v_i = g_i^{\sharp}(\overline{v}_i)$, and let $u'_i = f(\overline{u}_i)$ and $v'_i = g(\overline{v}_i)$. By definition of $\text{DP}(\mathcal{R}, \mu)$, $u'_i = l_i$ and v'_i is a replacing subterm of the right-hand side r_i of a rule $l_i \rightarrow r_i$ in \mathcal{R} (i.e., $r_i = C_i[v'_i]_{p_i}$ and $p_i \in \text{Pos}^{\mu}(r)$). Taking into account the definition of μ^{\sharp} , we obtain an infinite μ -rewrite sequence

$$\sigma(u'_1) \xrightarrow{\epsilon}_{\mathcal{R}, \mu} \sigma(C_1[v'_1]_{p_1}) \xrightarrow{>\epsilon}_{\mathcal{R}, \mu} \sigma(C_1[u'_2]_{p_1}) \xrightarrow{\epsilon}_{\mathcal{R}, \mu} \sigma(C_1[C_2[v'_2]_{p_2}]_{p_1}) \xrightarrow{>\epsilon}_{\mathcal{R}, \mu} \cdots$$

which contradicts μ -termination of \mathcal{R} . \square

Note that Theorem 2 does not impose any restriction on the TRSs for guaranteeing completeness. Unfortunately, without μ -conservativity, the use of $\text{DP}(\mathcal{R}, \mu)$ does not guarantee the necessary correctness.

Example 6 Consider the TRS \mathcal{R} :

$$\begin{aligned} a &\rightarrow f(b, a) \\ f(b, y) &\rightarrow f(y, c) \end{aligned}$$

and $\mu(f) = \{1\}$. Then, $\text{DP}(\mathcal{R}, \mu)$ is:

$$\begin{aligned} A &\rightarrow F(b, a) \\ F(b, y) &\rightarrow F(y, c) \end{aligned}$$

(and $\mu^\sharp(F) = \{1\}$) which does not induce any infinite chain of dependency pairs. However, \mathcal{R} is not μ -terminating:

$$\underline{a} \hookrightarrow \underline{f(b, a)} \hookrightarrow f(\underline{a}, c) \hookrightarrow f(\underline{f(b, a)}, c) \hookrightarrow \dots$$

Note that \mathcal{R} is not μ -conservative: $\text{Var}^\mu(f(y, c)) = \{y\}$, but $\text{Var}^\mu(f(b, y)) = \emptyset$.

Remark 1 Example 6 also shows that a mix which computes dependency pairs like in $\text{DP}(\mathcal{R}, \mu)$ for conservative rules and dependency pairs like in $\overline{\text{DP}}(\mathcal{R}, \mu)$ for non-conservative rules does not yield a correct approach.

Using Proposition 2, we can prove the following.

Theorem 3 (Restricted correctness) Let \mathcal{R} be a TRS and $\mu \in M_{\mathcal{R}}$. If \mathcal{R} is μ -conservative and there is no infinite $(\mathcal{R}, \text{DP}(\mathcal{R}, \mu), \mu^\sharp)$ -chain, then \mathcal{R} is μ -terminating.

Example 7 (Continuing Example 5) Now, $\text{DP}(\mathcal{R}, \mu)$ is:

$$F(a) \rightarrow F(f(a))$$

with $\mu^\sharp(F) = \emptyset$. Since $\mu^\sharp(F) = \emptyset$, no infinite chain is possible now.

The following result expresses that, despite the lack of completeness or correctness derived from the use of $\overline{\text{DP}}(\mathcal{R}, \mu)$ or $\text{DP}(\mathcal{R}, \mu)$, the μ -termination of a given TRS \mathcal{R} is characterized by one of these two sets of CS-dependency pairs.

Theorem 4 (Characterization of μ -termination) Let \mathcal{R} be a TRS and $\mu \in M_{\mathcal{R}}$. For either $\mathcal{P} = \text{DP}(\mathcal{R}, \mu)$ or $\mathcal{P} = \overline{\text{DP}}(\mathcal{R}, \mu)$ we have that \mathcal{R} is μ -terminating if and only there is no infinite $(\mathcal{R}, \mathcal{P}, \mu^\sharp)$ -chain.

PROOF. Assume that there is a TRS \mathcal{R} and $\mu \in M_{\mathcal{R}}$ such that neither $\text{DP}(\mathcal{R}, \mu)$ nor $\overline{\text{DP}}(\mathcal{R}, \mu)$ characterizes the μ -termination of \mathcal{R} . We consider two cases:

1. If \mathcal{R} is μ -terminating, then there is an infinite $(\mathcal{R}, \text{DP}(\mathcal{R}, \mu), \mu^\sharp)$ -chain which contradicts Theorem 2.
2. If \mathcal{R} is not μ -terminating, then there is no infinite $(\mathcal{R}, \overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp)$ -chain. This contradicts Theorem 1.

□

So, the problem is *deciding* which is the appropriate set of context-sensitive dependency pairs which should be considered for a given CS-TRS. The following example tries to justify that, in general, this is *not* possible.

Example 8 Consider the TRS \mathcal{R} [Zan97, Example 5]:

```
f(X) -> if(X,c,f(true))
if(true,X,Y) -> X
if(false,X,Y) -> Y
```

with $\mu(\text{if}) = \{1, 2\}$. As shown by Zantema, \mathcal{R} is μ -terminating. However, $\overline{\text{DP}}(\mathcal{R}, \mu)$ is:

```
F(X) -> IF(X,c,f(true))
F(X) -> F(true)
```

with $\mu^\sharp(\text{F}) = \{1\}$. We have an infinite chain

$$\text{F}(\text{true}) \xrightarrow{\epsilon}_{\overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp} \text{F}(\text{true}) \xrightarrow{\epsilon}_{\overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp} \dots$$

of dependency pairs which shows that, according to Theorem 4, $\text{DP}(\mathcal{R}, \mu)$

```
F(X) -> IF(X,c,f(true))
```

which admits no infinite chain, should be used instead to conclude μ -termination.

Now consider the following small variation \mathcal{R}' of \mathcal{R} :

```
f(X) -> if(X,c,f(false))
if(true,X,Y) -> X
if(false,X,Y) -> Y
```

with the same replacement map μ . Note that $\overline{\text{DP}}(\mathcal{R}', \mu)$:

```
F(X) -> IF(X,c,f(false))
F(X) -> F(false)
```

induces a similar infinite chain

$$\text{F}(\text{false}) \xrightarrow{\epsilon}_{\overline{\text{DP}}(\mathcal{R}', \mu), \mu^\sharp} \text{F}(\text{false}) \xrightarrow{\epsilon}_{\overline{\text{DP}}(\mathcal{R}', \mu), \mu^\sharp} \dots$$

whereas $\text{DP}(\mathcal{R}', \mu)$ again does not admit any infinite $(\mathcal{R}, \text{DP}(\mathcal{R}', \mu), \mu^\sharp)$ -chain. But \mathcal{R}' is not μ -terminating now!

Example 8 shows that no general ‘blind’ criterion is able to give a correct and complete characterization of μ -termination by using context-sensitive dependency pairs: in Example 8, first checking for infinite $(\mathcal{R}, \overline{\text{DP}}(\mathcal{R}, \mu), \mu^\sharp)$ -chains and then for infinite $(\mathcal{R}, \text{DP}(\mathcal{R}, \mu), \mu^\sharp)$ -chains yields the *same* result (yes / no) for two different CS-TRSs having different μ -termination behavior. Then, an explicit consideration of semantic (i.e., reducibility) properties of CS-TRSs is necessary. This suggests that (in sharp contrast to the unrestricted case) implementing a general procedure for deciding how to treat each concrete CS-TRS in this framework is not possible. The following fact is obvious from the definitions of $\overline{\text{DP}}(\mathcal{R}, \mu)$ and $\text{DP}(\mathcal{R}, \mu)$.

Proposition 3 Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. If for all $l \rightarrow r \in R$ and term u such that $r \succeq_{\mu} u$ and $\text{root}(u) \in \mathcal{D}$ there is no s such that $\text{root}(s) \in \mathcal{D}$ and s is a non-replacing subterm of u ($u \succeq - \succeq_{\mu} s$), then $\overline{\text{DP}}(\mathcal{R}, \mu) = \text{DP}(\mathcal{R}, \mu)$.

As a corollary of the previous results, we have.

Corollary 1 (Restricted characterization of μ -termination) *Let \mathcal{R} be a TRS and $\mu \in M_{\mathcal{R}}$. Assume that \mathcal{R} is either μ -conservative or it satisfies the condition in Proposition 3. Then, \mathcal{R} is μ -terminating if and only there is no infinite $(\mathcal{R}, \text{DP}(\mathcal{R}, \mu), \mu^\sharp)$ -chain.*

5 Context-sensitive dependency graph

As noticed by Arts and Giesl, the analysis of infinite sequences of dependency pairs can be made by looking at (the cycles \mathfrak{C} of) the *dependency graph* associated to the TRS \mathcal{R} . The nodes of the dependency graph are the dependency pairs in $\text{DP}(\mathcal{R})$; there is an arc from a dependency pair $u \rightarrow v$ to a dependency pair $u' \rightarrow v'$ if there are substitutions σ and θ such that $\sigma(v) \rightarrow_{\mathcal{R}}^* \theta(u')$. In the *context-sensitive dependency graph*, there is an arc from a dependency pair $u \rightarrow v$ to a dependency pair $u' \rightarrow v'$ if there are substitutions σ and θ such that $\sigma(v) \hookrightarrow_{\mathcal{R}, \mu^\sharp}^* \theta(u')$. Here, the use of μ^\sharp (which restricts reductions on the arguments of the dependency pair symbols f^\sharp) is essential: given a set of dependency pairs associated to a CS-TRS (\mathcal{R}, μ) , we have less arcs between them due to the presence of such replacement restrictions.

Example 9 *Consider the CS-TRS in Example 1. $\overline{\text{DP}}(\mathcal{R}, \mu) = \text{DP}(\mathcal{R}, \mu)$ is:*

$$F(\mathbf{a}, \mathbf{b}, \mathbf{X}) \rightarrow F(\mathbf{X}, \mathbf{X}, \mathbf{X})$$

with $\mu^\sharp(F) = \{3\}$. Although the dependency graph contains a cycle (due to $\sigma(F(\mathbf{X}, \mathbf{X}, \mathbf{X})) \rightarrow^ \sigma(F(\mathbf{a}, \mathbf{b}, \mathbf{Y}))$ for $\sigma(X) = \sigma(Y) = \mathbf{c}$), the CS-dependency graph contains no cycle because it is not possible to μ^\sharp -reduce $\theta(F(\mathbf{X}, \mathbf{X}, \mathbf{X}))$ into $\theta(F(\mathbf{a}, \mathbf{b}, \mathbf{Y}))$ for any substitution θ (due to $\mu^\sharp(F) = \{3\}$).*

As noticed by Arts and Giesl, the presence of an infinite chain of dependency pairs correspond to a cycle in the dependency graph (but not vice-versa). Thus, as a corollary of Theorem 1, we have:

Corollary 2 *Let \mathcal{R} be a TRS and $\mu \in M_{\mathcal{R}}$. If the CS-dependency graph built from $\overline{\text{DP}}(\mathcal{R}, \mu)$ contains no cycle, then \mathcal{R} is μ -terminating.*

According to this, and continuing Example 9, we conclude the μ -termination of \mathcal{R} in Example 1. Of course, the interesting situation is when the dependency graph contains cycles. The following definition borrows [HM04, Definition 8] and will be used later.

Definition 3 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $\mathfrak{C} \subseteq \overline{\text{DP}}(\mathcal{R}, \mu)$ be a cycle. An infinite μ^\sharp -rewrite sequence in $\mathcal{R} \cup \mathfrak{C}$ of the form*

$$t_1^\sharp \hookrightarrow_{\mathcal{R}, \mu^\sharp}^* s_1 \hookrightarrow_{\mathfrak{C}, \mu^\sharp} t_2 \hookrightarrow_{\mathcal{R}, \mu^\sharp}^* s_2 \hookrightarrow_{\mathfrak{C}, \mu^\sharp} t_3 \hookrightarrow_{\mathcal{R}, \mu^\sharp}^* \dots$$

with $t_1 \in \mathcal{T}_{\infty, \mu}$ is called \mathfrak{C} -minimal if all rules in \mathfrak{C} are applied infinitely often.

Following Hirokawa and Middeldorp, *proving μ -termination boils down to proving the absence of \mathfrak{C} -minimal μ -rewrite sequences, for any cycle \mathfrak{C} in the CS-dependency graph.*

5.1 Estimating the CS-dependency graph

In general, the dependency graph of a TRS is *not* computable and we need to use some approximation of it. Following [AG00], we describe how to approximate the CS-dependency graph of a CS-TRS (\mathcal{R}, μ) . Let CAP^μ be given as follows: let D be a set of defined symbols (in our context, $D = \mathcal{D} \cup \mathcal{D}^\sharp$):

$$\begin{aligned} \text{CAP}^\mu(x) &= x \text{ if } x \text{ is a variable} \\ \text{CAP}^\mu(f(t_1, \dots, t_k)) &= \begin{cases} y & \text{if } f \in D \\ f([t_1]_1^f, \dots, [t_k]_1^f) & \text{otherwise} \end{cases} \end{aligned}$$

where y is intended to be a new, fresh variable which has not yet been used and given a term s , $[s]_i^f = \text{CAP}^\mu(s)$ if $i \in \mu(f)$ and $[s]_i^f = s$ if $i \notin \mu(f)$. Let REN^μ given by: $\text{REN}^\mu(x) = y$ if x is a variable and $\text{REN}^\mu(f(t_1, \dots, t_k)) = f([t_1]_1^f, \dots, [t_k]_k^f)$ for every k -ary symbol f , where given a term $s \in \mathcal{T}^\sharp(\mathcal{F}, \mathcal{X})$, $[s]_i^f = \text{REN}^\mu(s)$ if $i \in \mu(f)$ and $[s]_i^f = s$ if $i \notin \mu(f)$. Then, we have an arc from $u_i \rightarrow v_i$ to $u_j \rightarrow v_j$ if $\text{REN}^\mu(\text{CAP}^\mu(v_i))$ and u_j unify; following [AG00], we say that v_i and u_j are μ -connectable. The following result whose proof is similar to that of [AG00, Theorem 21] (we only need to take into account the replacement restrictions indicated by the replacement map μ) formalizes the correctness of this approach.

Proposition 4 *Let (\mathcal{R}, μ) be a CS-TRS. If there is an arc from $u \rightarrow v$ to $u' \rightarrow v'$ in the CS-dependency graph, then v and u' are μ -connectable.*

Example 10 *(Continuing Ex. 9) Since $\text{REN}^{\mu^\sharp}(\text{CAP}^{\mu^\sharp}(\mathbf{F}(\mathbf{X}, \mathbf{X}, \mathbf{X}))) = \mathbf{F}(\mathbf{X}, \mathbf{X}, \mathbf{Z})$ and $\mathbf{F}(\mathbf{a}, \mathbf{b}, \mathbf{Y})$ do not unify, we conclude (and this can be easily implemented) that the CS-dependency graph for the CS-TRS (\mathcal{R}, μ) in Example 1 contains no cycles.*

6 Subterm criterion

In [HM04], Hirokawa and Middeldorp introduce a very interesting *subterm criterion* which permits to ignore certain cycles of the dependency graph.

Definition 4 ([HM04]) *Let \mathcal{R} be a TRS and $\mathfrak{C} \subseteq \text{DP}(\mathcal{R})$ such that every dependency pair symbol in \mathfrak{C} has positive arity. A simple projection for \mathfrak{C} is a mapping π that assigns to every k -ary dependency pair symbol f^\sharp in \mathfrak{C} an argument position $i \in \{1, \dots, k\}$. The mapping that assigns to every term $f^\sharp(t_1, \dots, t_k) \in \mathcal{T}^\sharp(\mathcal{F}, \mathcal{X})$ with f^\sharp a dependency pair symbol in \mathcal{R} its argument position $\pi(f^\sharp)$ is also denoted by π .*

In the following result, for a simple projection π and $\mathfrak{C} \subseteq \overline{\text{DP}}(\mathcal{R}, \mu)$, we let $\pi(\mathfrak{C}) = \{\pi(u \rightarrow v) \mid u \rightarrow v \in \mathfrak{C}\}$, where $\pi(u \rightarrow v) = \pi(u) \rightarrow \pi(v)$ if $u \rightarrow v$ is a dependency pair. Note that $u, v \in \mathcal{T}^\sharp(\mathcal{F}, \mathcal{X})$, but $\pi(u), \pi(v) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$.

Theorem 5 *Let \mathcal{R} be a TRS and $\mu \in M_{\mathcal{R}}$. Let \mathfrak{C} be a cycle in $\overline{\text{DG}}(\mathcal{R})$. If there exists a simple projection π for \mathfrak{C} such that $\pi(\mathfrak{C}) \subseteq \succeq_{\mu}$ and $\pi(\mathfrak{C}) \cap \triangleright_{\mu} \neq \emptyset$, then there are no minimal \mathfrak{C} μ^{\sharp} -rewrite sequences.*

PROOF. The proof is completely analogous to that of [HM04, Theorem 11]. The only difference is that we need to deal with the μ -subterm relation \succeq_{μ} ; this is because, we need to use the following commutation property: $\triangleright_{\mu} \circ \hookrightarrow_{\mathcal{R}, \mu} \subseteq \hookrightarrow_{\mathcal{R}, \mu} \circ \triangleright_{\mu}$ which does not hold if \triangleright is used instead. \square

Example 11 *Consider the CS-TRS (\mathcal{R}, μ) in Example 3. The dependency pair*

$$\text{SEL}(\mathfrak{s}(\mathbf{N}), \mathbf{X}:\mathbf{XS}) \rightarrow \text{SEL}(\mathbf{N}, \mathbf{XS})$$

induces a cycle. However, with $\pi(\text{SEL}) = 1$, we have that

$$\pi(\text{SEL}(\mathfrak{s}(\mathbf{N}), \mathbf{X}:\mathbf{XS})) = \mathfrak{s}(\mathbf{N}) \triangleright_{\mu} \mathbf{N} = \pi(\text{SEL}(\mathbf{N}, \mathbf{XS}))$$

Hence, by Theorem 5 we conclude the μ -termination of \mathcal{R} .

Restricting the attention to *replacing* rather than arbitrary subterms is essential to achieve a correct technique: assume that the third rule of \mathcal{R} in Example 3 is

$$\mathfrak{sel}(\mathfrak{s}(\mathbf{N}), \mathbf{X}:\mathbf{XS}) \rightarrow \mathfrak{sel}(\mathfrak{s}(\mathbf{N}), \mathbf{XS})$$

and use the same replacement map μ . The system is not μ -terminating now, but the application of Hirokawa and Middeldorp's criterion (using $\pi(\text{SEL}) = 2$ and \triangleright instead of \triangleright_{μ}) to:

$$\text{SEL}(\mathfrak{s}(\mathbf{N}), \mathbf{X}:\mathbf{XS}) \rightarrow \text{SEL}(\mathfrak{s}(\mathbf{N}), \mathbf{XS})$$

leads to wrongly conclude the μ -termination of the system.

Example 12 *Consider the CS-TRS (\mathcal{R}, μ) in Example 2. The CS-dependency graph of \mathcal{R} consists of seven pairs with only four cycles:*

$$(C1) \ \{\text{SQR}(\mathfrak{s}(X)) \rightarrow \text{SQR}(X)\}$$

$$(C2) \ \{\text{DBL}(\mathfrak{s}(X)) \rightarrow \text{DBL}(X)\}$$

$$(C3) \ \{\text{ADD}(\mathfrak{s}(X), Y) \rightarrow \text{ADD}(X, Y)\}$$

$$(C4) \ \{\text{HALF}(\mathfrak{s}(\mathfrak{s}(X))) \rightarrow \text{HALF}(X)\}$$

Each of them can be removed by using the subterm criterion. Thus, by Theorem 5, the system is μ -terminating.

As remarked by Hirokawa and Middeldorp, the practical use of Theorem 5 concerns the so-called *strongly connected components* (SCCs) of the dependency graph, rather than the cycles themselves (which are exponentially many) [HM04, HM05]. A strongly connected component in the (CS-)dependency graph is a maximal cycle, i.e., it is not contained in any other cycle. According to Hirokawa and Middeldorp, the adaptation of the subterm criterion to the CS-dependency graph recursively applies as follows: when considering an SCC \mathfrak{C} , we *remove* from \mathfrak{C} those pairs $u \rightarrow v$ satisfying $\pi(u) \triangleright_{\mu} \pi(v)$. Then, we recompute the

SCCs with the remaining pairs in the CS-dependency graph and start again (see [HM05, Section 4]).

For the cycles left after the removal performed by means of the subterm criterion, the absence of infinite chains is checked by finding, for those cycles, a *reduction pair* (\succeq, \sqsupset) . In our setting, we can relax the monotonicity requirements on the first component of the reduction pair (\succeq, \sqsupset) . We will use μ -reduction pairs (\succeq_μ, \sqsupset) where \succeq_μ is a stable and μ -monotonic preordering which is compatible with the well-founded and stable ordering \sqsupset , i.e., $\succeq_\mu \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \succeq_\mu \subseteq \sqsupset$. On the other hand, the use of *argument filterings*, which is standard in the current formulations of the dependency pairs method, also adapts without changes to the context-sensitive setting. Here, an argument filtering π for a signature \mathcal{F} is a mapping that assigns to every k -ary function symbol $f \in \mathcal{F}$ an argument position $i \in \{1, \dots, k\}$ or a (possibly empty) list $[i_1, \dots, i_m]$ of argument positions with $1 \leq i_1 < \dots < i_m \leq k$. The signature \mathcal{F}_π consists of all function symbols f such that $\pi(f)$ is some list $[i_1, \dots, i_m]$, where in \mathcal{F}_π , the arity of f is m . Every argument filtering induces a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{X})$ to $\mathcal{T}(\mathcal{F}_\pi, \mathcal{X})$, also denoted by π :

$$\pi(t) = \begin{cases} t & \text{if } t \text{ is a variable} \\ \pi(t_i) & \text{if } t = f(t_1, \dots, t_k) \text{ and } \pi(f) = i \\ f(\pi(t_{i_1}), \dots, \pi(t_{i_m})) & \text{if } t = f(t_1, \dots, t_k) \text{ and } \pi(f) = [i_1, \dots, i_m] \end{cases}$$

The proof of the following Theorem is completely analogous to standard ones (see, e.g., [HM04, Theorem 18]).

Theorem 6 (Use of the CS-dependency graph) *Let \mathcal{R} be a TRS and $\mu \in M_{\mathcal{R}}$. Let \mathfrak{C} be a cycle in the CS-dependency graph. If there is an argument filtering π and a μ -reduction pair (\succeq_μ, \sqsupset) such that $\pi(\mathcal{R}) \subseteq \succeq_\mu$, $\pi(\mathfrak{C}) \subseteq (\succeq_\mu \cup \sqsupset)$ and $\pi(\mathfrak{C}) \cap \sqsupset \neq \emptyset$, then there is no \mathfrak{C} -minimal μ^\sharp -rewrite sequence.*

Example 13 *Consider the CS-TRS (\mathcal{R}, μ) in Example 4. Since \mathcal{R} is μ -conservative, by Theorem 3, we only need to consider dependency pairs in $\text{DP}(\mathcal{R}, \mu)$:*

$$\begin{aligned} \mathbf{G}(\mathbf{X}) &\rightarrow \mathbf{H}(\mathbf{X}) \\ \mathbf{H}(\mathbf{d}) &\rightarrow \mathbf{G}(\mathbf{c}) \end{aligned}$$

Note that the dependency graph contains a single cycle including both of them. We can prove the μ -termination of \mathcal{R} by using the following polynomial interpretation over the natural numbers:

$$\begin{aligned} [\mathbf{g}](x) &= 0 & [\mathbf{c}] &= 1 & [\mathbf{G}](x) &= (1-x)^2 \\ [\mathbf{h}](x) &= 0 & [\mathbf{d}] &= 0 & [\mathbf{H}](x) &= (1-x)^2 \end{aligned}$$

which induces a μ -reduction pair $(\succeq, >)$ which is compatible with \mathcal{R} and the cycle in $\text{DP}(\mathcal{R}, \mu)$ in the sense of Theorem 6.

Remark 2 *Note that the conditions of use of Theorem 6 for context-sensitive dependency graphs are almost the same than for the standard dependency graph.*

The only difference is that the preordering in a μ -reduction pair is μ -monotonic rather than monotonic. No restriction on the considered argument filterings (regarding the underlying replacement map μ) is considered.

7 Experiments

We have implemented the techniques described in the previous sections as part of the tool MU-TERM [Luc04a]. Following Theorem 1, given a CS-TRS (\mathcal{R}, μ) , the tool automatically generates the dependency pairs in $\overline{\text{DP}}(\mathcal{R}, \mu)$ (or in $\text{DP}(\mathcal{R}, \mu)$ if \mathcal{R} is μ -conservative, see Theorem 3) and tries to prove that no cycle \mathfrak{C} in the (estimated) context-sensitive dependency graph induces an infinite \mathfrak{C} -minimal μ^\sharp -rewrite sequence by using Theorems 5 and 6 and eventually Corollary 2. Since our current implementation is based on the use of polynomial orderings, we actually do not compute any argument filtering function when using Theorem 6; it is well-known that they are somehow implicit in the computation of the polynomial interpretations (see, e.g., [AG00]).

We have considered the examples in the following WWW page:

<http://www.dsic.upv.es/~slucas/csr/termination/examples>

which collects almost all published examples of non-terminating TRSs which can be proved μ -terminating for concrete replacement maps μ : 42 examples (out from around 20 different references). We have used our prototype implementation of the CS-dependency pairs technique to obtain a comparison of different techniques for proving termination of CSR. We consider the CSRPO [BLR02] (although it is not implemented yet), the polynomial orderings generated by MU-TERM according to [Luc05], and the transformations introduced by Giesl and Middeldorp [GM04] and Zantema [Zan97] which, according to the analysis in [Luc06], have the most successful behavior among all existing ones. The following table summarizes our results:

	CS-DPs	CSRPO	POL	GM	Z
Automatic:	31	0	19	20	23
Total:	33	24	25	20	23

We have also tried all such examples by using AProVE [GTSF04], which is able to prove termination of CSR by using the aforementioned transformations¹. Our results are also really good: as indicated above, 33 examples were proved by using the CS-DP approach *alone* (31 of them automatically by using MU-TERM). AProVE succeeded on 32 examples by using different transformations each time and then using the impressive amount of implemented techniques for proving termination of rewriting, see [GTSF04]. This is in sharp contrast with the simplicity of our first prototype for the CS-DP approach (just the techniques described in this paper!). Furthermore, we note that proofs using AProVE are

¹We have used the Web-version: <http://aprove.informatik.rwth-aachen.de>, which accepts CS-TRSs in the TPDB format, see <http://www.lri.fr/~marche/tpdb/format.html>.

much time consuming than using MU-TERM. The total time of MU-TERM on the 31 successfully proved examples was 1.98 seconds. The 32 successful proofs with AProVE took 124 seconds. Therefore, the average time of MU-TERM is 0.06 seconds versus the average of 3.88 seconds for AProVE.

8 Conclusions

We have shown how to use dependency pairs in proofs of termination of *CSR*. The implementation and practical use of the developed techniques yield a novel and powerful framework which dramatically improves the current state-of-the-art of methods for proving termination of *CSR*. Some interesting differences arise which can be summarized as follows: in contrast to the standard dependency pairs framework, where there is a set of dependency pairs $\text{DP}(\mathcal{R})$ which characterizes both termination and non-termination of \mathcal{R} , in *CSR* there are *two* sets $\overline{\text{DP}}(\mathcal{R}, \mu)$ and $\text{DP}(\mathcal{R}, \mu)$ (where $\text{DP}(\mathcal{R}, \mu) \subseteq \overline{\text{DP}}(\mathcal{R}, \mu) \subseteq \text{DP}(\mathcal{R})$) of dependency pairs associated to a CS-TRS (\mathcal{R}, μ) which are relevant to analyze its termination behavior. Ensuring that there is no infinite chain of dependency pairs from $\overline{\text{DP}}(\mathcal{R}, \mu)$ guarantees that \mathcal{R} is μ -terminating (Theorem 1) but having an infinite chain does *not* implies the non- μ -termination of \mathcal{R} (Example 5). On the other hand, finding an infinite chain of dependency pairs from $\text{DP}(\mathcal{R}, \mu)$ guarantees that \mathcal{R} is *not* μ -terminating (Theorem 2), but ensuring the absence of infinite chains does *not* implies the μ -termination of \mathcal{R} (Example 6). Still, given a CS-TRS, we know that its μ -termination behavior is completely determined by *one* of these two sets (Theorem 4), but in general it is not decidable which is appropriate one (Example 8). Fortunately, in some restricted cases, we know how to make this choice thanks to some easy syntactic characterizations (Corollary 1).

Regarding the practical use of the CS-dependency pairs in proofs of termination of *CSR*, we have shown how to build and use the corresponding CS-dependency graph to either prove that there are cycles which do not need to be considered at all (Theorem 5) or to prove that the rules of the TRS and the cycles in the CS-dependency graph are compatible with some reduction pair (Theorem 6). We have implemented these ideas as part of the termination tool MU-TERM; after a thorough comparison with other techniques for proving termination of *CSR*, in particular those implemented by other termination tools like AProVE (see Section 7), we can conclude that the CS-dependency pairs can play in *CSR* the role than dependency pairs play in rewriting.

The research in this paper leaves an interesting problem to be further investigated: the definition of decidable sets of CS-TRSs (\mathcal{R}, μ) for which it is possible to say the set $(\overline{\text{DP}}(\mathcal{R}, \mu)$ or $\text{DP}(\mathcal{R}, \mu))$ of dependency pairs which characterizes the μ -termination of \mathcal{R} . Interestingly, most of the examples which cannot be proved μ -terminating by using CS-DPs seem to fail due to a ‘bad’ choice of the set of CS-dependency pairs. For instance, the following CS-TRS [Zan97, Introduction]:

```

fact(X) -> if(zero(X),s(0),prod(X,fact(p(X))))
add(0,X) -> X
add(s(X),Y) -> s(add(X,Y))
prod(0,X) -> 0
prod(s(X),Y) -> add(Y,prod(X,Y))
if(true,X,Y) -> X
if(false,X,Y) -> Y
zero(0) -> true
zero(s(X)) -> false
p(s(X)) -> X

```

with $\mu(\text{if}) = \{1\}$ and $\mu(f) = \{1, \dots, k\}$ for any other k -ary symbols f was claimed to be terminating by Zantema, but no proof (or disproof) of this claim is known yet. The analysis of $\overline{\text{DP}}(\mathcal{R}, \mu)$ leads to consider the following three cycles:

```

(C1) { FACT(X) -> FACT(p(X)) }
(C2) { ADD(s(X),Y) -> ADD(X,Y) }
(C3) { PROD(s(X),Y) -> PROD(X,Y) }

```

Cycles $C2$ and $C3$ can be removed by using the subterm criterion, but nothing can be done with $C1$. In fact, $C1$ itself induces an infinite chain (without any real μ -rewriting step with \mathcal{R} !). No result in this paper allows us to conclude that \mathcal{R} is non- μ -terminating, but we know that the μ -termination of \mathcal{R} cannot be proved by using $\overline{\text{DP}}(\mathcal{R}, \mu)$. On the other hand, the dependency pair in $C1$ does *not* belong to $\text{DP}(\mathcal{R}, \mu)$. Hence, if we could ensure that $\text{DP}(\mathcal{R}, \mu)$ provides the desired characterization of μ -termination, proving it would be immediate.

There are many other aspects of the dependency pairs framework which are also worth to be considered and eventually extended to *CSR* (e.g., narrowing refinements, modularity issues, innermost computations, usable rules, ...). These aspects provide an interesting subject for future work.

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